

Type Reconstruction and Polymorphism

Week 9

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Type Checking and Type Reconstruction

We now come to the question of type checking and type reconstruction.

Type checking: Given Γ , t and T , check whether $\Gamma \vdash t : T$

Type reconstruction: Given Γ and t , find a type T such that $\Gamma \vdash t : T$

Type checking and reconstruction seem difficult since parameters in lambda calculus do not carry their types with them.

Type reconstruction also suffers from the problem that a term can have many types.

Idea: : We construct all type derivations in parallel, reducing type reconstruction to a unification problem.

From Judgements to Equations

$TP : \text{Judgement} \rightarrow \text{Equations}$

$TP(\Gamma \vdash t : T) =$

case t of

$x \quad : \quad \{\Gamma(x) \hat{=} T\}$

$\lambda x.t' \quad : \quad \text{let } a, b \text{ fresh in}$

$\{(a \rightarrow b) \hat{=} T\} \cup$

$TP(\Gamma, x : a \vdash t' : b)$

$t \ t' \quad : \quad \text{let } a \text{ fresh in}$

$TP(\Gamma \vdash t : a \rightarrow T) \cup$

$TP(\Gamma \vdash t' : a)$

Constants

Constants are treated as variables in the initial environment.

However, we have to make sure we create a new instance of their type as follows:

$$\begin{aligned} \text{newInstance}(\forall a_1, \dots, a_n. S) = \\ \quad \text{let } b_1, \dots, b_n \text{ fresh in} \\ \quad [b_1/a_1, \dots, b_n/a_n] S \\ TP(\Gamma \vdash t : T) = \\ \quad \text{case } t \text{ of} \\ \quad \quad x \quad : \quad \{ \text{newInstance}(\Gamma(x)) \hat{=} T \} \\ \quad \quad \dots \end{aligned}$$

Soundness and Completeness I

Definition: In general, a type reconstruction algorithm $\mathcal{A}$ assigns to an environment Γ and a term t a set of types $\mathcal{A}(\Gamma, t)$.

The algorithm is **sound** if for every type $T \in \mathcal{A}(\Gamma, t)$ we can prove the judgement $\Gamma \vdash t : T$.

The algorithm is **complete** if for every provable judgement $\Gamma \vdash t : T$ we have that $T \in \mathcal{A}(\Gamma, t)$.

Theorem: TP is sound and complete. Specifically:

$$\Gamma \vdash t : T \quad \text{iff} \quad \exists \bar{b}. [T/a]EQNS$$

where

a is a new type variable

$$EQNS = TP(\Gamma \vdash t : a)$$

$$\bar{b} = tv(EQNS) \setminus tv(\Gamma)$$

Here, tv denotes the set of free type variables (of a term, and environment, an equation set).

Type Reconstruction and Unification

Problem: : Transform set of equations

$$\{T_i \hat{=} U_i\}_{i=1, \dots, m}$$

into equivalent substitution

$$\{a_j \hat{=} T'_j\}_{j=1, \dots, n}$$

where type variables do not appear recursively on their right hand sides (directly or indirectly). That is:

$$a_j \notin tv(T'_k) \quad \text{for } j = 1, \dots, n, k = j, \dots, n$$

Substitutions

A **substitution** s is an idempotent mapping from type variables to types which maps all but a finite number of type variables to themselves.

We often represent a substitution as a set of equations $a \hat{=} T$ with a not in $tv(T)$.

Substitutions can be generalized to mappings from types to types by defining

$$\begin{aligned} s(T \rightarrow U) &= sT \rightarrow sU \\ s(K[T_1, \dots, T_n]) &= K[sT_1, \dots, sT_n] \end{aligned}$$

Substitutions are idempotent mappings from types to types, i.e. $s(s(T)) = s(T)$. (why?)

The $\circ$ operator denotes composition of substitutions (or other functions): $(f \circ g) x = f(gx)$.

A Unification Algorithm

We present an incremental version of Robinson's algorithm (1965).

$$\begin{aligned} mgu & : (Type \hat{=} Type) \rightarrow Subst \rightarrow Subst \\ mgu(T \hat{=} U) s & = mgu'(sT \hat{=} sU) s \\ mgu'(a \hat{=} a) s & = s \\ mgu'(a \hat{=} T) s & = s \cup \{a \hat{=} T\} \quad \text{if } a \notin tv(T) \\ mgu'(T \hat{=} a) s & = s \cup \{a \hat{=} T\} \quad \text{if } a \notin tv(T) \\ mgu'(T \rightarrow T' \hat{=} U \rightarrow U') s & = (mgu(T' \hat{=} U') \circ mgu(T \hat{=} U)) s \\ mgu'(K[T_1, \dots, T_n] \hat{=} K[U_1, \dots, U_n]) s & \\ & = (mgu(T_n \hat{=} U_n) \circ \dots \circ mgu(T_1 \hat{=} U_1)) s \\ mgu'(T \hat{=} U) s & = error \quad \text{in all other cases} \end{aligned}$$

Soundness and Completeness of Unification

Definition: A substitution u is a **unifier** of a set of equations $\{T_i \hat{=} U_i\}_{i=1, \dots, m}$ if $uT_i = uU_i$, for all i . It is a **most general unifier** if for every other unifier u' of the same equations there exists a substitution s such that $u' = s \circ u$.

Theorem: Given a set of equations $EQNS$. If $EQNS$ has a unifier then $mgu\ EQNS\ \{\}$ computes the most general unifier of $EQNS$. If $EQNS$ has no unifier then $mgu\ EQNS\ \{\}$ fails.

From Judgements to Substitutions

$TP : \text{Judgement} \rightarrow \text{Subst} \rightarrow \text{Subst}$

$TP(\Gamma \vdash t : T) =$

case t of

x : **mgu**($\text{newInstance}(\Gamma x) \hat{=} T$)

$\lambda x.t'$: **let** t, u **fresh in**

mgu(($t \rightarrow u$) $\hat{=} T$) $\circ$

$TP(\Gamma, x : t \vdash t' : u)$

$t \rightarrow t'$: **let** t **fresh in**

$TP(\Gamma \vdash t : a \rightarrow T)$ $\circ$

$TP(\Gamma \vdash t' : a)$

Soundness and Completeness II

One can show by comparison with the previous algorithm:

Theorem: TP is sound and complete. Specifically:

$$\Gamma \vdash t : T \text{ iff } T = r(s(t))$$

where

t is a new type variable

$$s = TP(\Gamma \vdash t : t) \{ \}$$

r is a substitution on $tv(s\ t) \setminus tv(s\ \Gamma)$

Strong Normalization

Question: Can Ω be given a type?

$$\Omega = (\lambda x.xx)(\lambda x.xx) : ?$$

What about Y ?

Self-application is not typable!

In fact, we have more:

Theorem: (Strong Normalization) If $\vdash t : T$, then there is a value V such that $t \rightarrow^* V$.

Corollary: Simply typed lambda calculus is not Turing complete.

Polymorphism

In the simply typed lambda calculus, a term can have many types.

But a variable or parameter has only one type.

Example:

$$(\lambda x.xx)(\lambda y.y)$$

is untypable. But if we substitute actual parameter for formal, we obtain

$$(\lambda y.y)(\lambda y.y) : a \rightarrow a$$

Functions which can be applied to arguments of many types are called **polymorphic**.

Polymorphism in Programming

Polymorphism is essential for many program patterns.

Example: map

```
def map f xs =  
  if (isEmpty (xs)) nil  
  else cons (f (head xs)) (map (f, tail xs))  
...  
names: List[String]  
nums  : List[Int]  
...  
map toUpperCase names  
map increment nums
```

Without a polymorphic type for map one of the last two lines is always illegal!

Forms of Polymorphism

Polymorphism means “having many forms”.

Polymorphism also comes in several forms.

- **Universal polymorphism**, sometimes also called **generic types**: The ability to instantiate type variables.
- **Inclusion polymorphism**, sometimes also called **subtyping**: The ability to treat a value of a subtype as a value of one of its supertypes.
- **Ad-hoc polymorphism**, sometimes also called **overloading**: The ability to define several versions of the same function name, with different types.

We first concentrate on universal polymorphism.

Two basic approaches: **explicit** or **implicit**.

Explicit Polymorphism

We introduce a polymorphic type $\forall a.T$, which can be used just as any other type.

We then need to make introduction and elimination of $\forall$'s explicit.

Typing rules:

$$\begin{array}{cc} (\forall E) \frac{\Gamma \vdash t : \forall a.T}{\Gamma \vdash t[U] : [U/a]T} & (\forall I) \frac{\Gamma \vdash t : T}{\Gamma \vdash \Lambda a.t : \forall a.T} \end{array}$$

We also need to give all parameter types, so programs become verbose.

Example:

```
def map [a][b] (f: a -> b) (xs: List[a]) =  
  if (isEmpty [a] (xs)) nil [a]  
  else cons [b] (f (head [a] xs)) (map [a][b] (f, tail [a] xs))  
...  
names: List[String]  
nums : List[Int]  
...  
map [String] [String] toUpperCase names  
map [Int] [Int] increment nums
```

Implicit Polymorphism

Implicit polymorphism does not require annotations for parameter types or type instantiations.

Idea: In addition to types (as in simply typed lambda calculus), we have a new syntactic category of **type schemes**. Syntax:

$$\text{Type Scheme } S ::= T \mid \forall a. S$$

Type schemes are not fully general types; they are used only to type named values, introduced by a `val` construct.

The resulting type system is called the **Hindley/Milner system**, after its inventors. (The original treatment uses `let ... in ...` rather than `val ... ; ...`).

Hindley/Milner Typing rules

$$(\text{VAR}) \quad \Gamma, x : \textcolor{red}{S}, \Gamma' \vdash x : \textcolor{red}{S} \quad (x \notin \text{dom}(\Gamma'))$$

$$(\forall\text{E}) \quad \frac{\Gamma \vdash t : \textcolor{red}{\forall a}.T}{\Gamma \vdash t : [U/a]T} \quad (\forall\text{I}) \quad \frac{\Gamma \vdash t : T \quad a \notin \text{tv}(\Gamma)}{\Gamma \vdash t : \textcolor{red}{\forall a}.T}$$

$$(\text{LET}) \quad \frac{\Gamma \vdash t : \textcolor{red}{S} \quad \Gamma, x : \textcolor{red}{S} \vdash t' : T}{\Gamma \vdash \textbf{let } x = t \textbf{ in } t' : T}$$

The other two rules are as in simply typed lambda calculus:

$$(\rightarrow\text{I}) \quad \frac{\Gamma, x : T \vdash t : U}{\Gamma \vdash \lambda x. t : T \rightarrow U} \quad (\rightarrow\text{E}) \quad \frac{\Gamma \vdash M : T \rightarrow U \quad \Gamma \vdash N : T}{\Gamma \vdash M N : U}$$

Hindley/Milner in Programming Languages

Here is a formulation of the map example in the Hindley/Milner system.

```
let map =  $\lambda f. \lambda xs$  in
  if (isEmpty (xs)) nil
  else cons (f (head xs)) (map (f, tail xs))
...
// names: List[String]
// nums  : List[Int]
// map   :  $\forall a. \forall b. (a \rightarrow b) \rightarrow \text{List}[a] \rightarrow \text{List}[b]$ 
...
map toUpperCase names
map increment nums
```

Limitations of Hindley/Milner

Hindley/Milner still does not allow parameter types to be polymorphic.
I.e.

$$(\lambda x.xx)(\lambda y.y)$$

is still ill-typed, even though the following is well-typed:

$$\textit{let } id = \lambda y.y \textit{ in } id \ id$$

With explicit polymorphism the expression could be completed to a well-typed term:

$$(\Lambda a.\lambda x : (\forall a : a \rightarrow a).x[a \rightarrow a](x[a]))(\Lambda b.\lambda y.y)$$

The Essence of let

We regard

let $x = t$ *in* t'

as a shorthand for

$[t/x]t'$

We use this equivalence to get a revised Hindley/Milner system.

Definition: Let HM' be the type system that results if we replace rule (LET) from the Hindley/Milner system HM by:

$$(\text{LET}') \frac{\Gamma \vdash t : T \quad \Gamma \vdash [t/x]t' : U}{\Gamma \vdash \text{let } x = t \text{ in } t' : U}$$

Theorem: $\Gamma \vdash_{HM} t : S$ iff $\Gamma \vdash_{HM'} t : S$

The theorem establishes the following connection between the Hindley/Milner system and the simply typed lambda calculus F_1 :

Corollary: Let t^* be the result of expanding all *let*'s in t according to the rule

$$\textit{let } x = t \textit{ in } t' \rightarrow [t/x]t'$$

Then

$$\Gamma \vdash_{HM} t : T \Rightarrow \Gamma \vdash_{F_1} t^* : T$$

Furthermore, if every *let*-bound name is used at least once, we also have the reverse:

$$\Gamma \vdash_{F_1} t^* : T \Rightarrow \Gamma \vdash_{HM} t : T$$

Principal Types

Definition: A type T is a **generic instance** of a type scheme $S = \forall\alpha_1 \dots \forall\alpha_n.T'$ if there is a substitution s on $\alpha_1, \dots, \alpha_n$ such that $T = sT'$. We write in this case $S \leq T$.

Definition: A type scheme S' is a generic instance of a type scheme S iff for all types T

$$S' \leq T \Rightarrow S \leq T$$

We write in this case $S \leq S'$.

Definition: A type scheme S is **principal** (or: **most general**) for Γ and t iff

- $\Gamma \vdash t : S$
- $\Gamma \vdash t : S'$ implies $S \leq S'$

Definition: A type system TS has the **principal typing property** iff, whenever $\Gamma \vdash_{TS} t : S$ then there exists a principal type scheme for Γ and t .

Theorem:

1. HM' without **let** has the p.t.p.
2. HM' with **let** has the p.t.p.
3. HM has the p.t.p.

Proof sketch: (1.): Use type reconstruction result for the simply typed lambda calculus. (2.): Expand all **let**'s and apply (1.). (3.): Use equivalence between HM and HM' .

These observations could be used to come up with a type reconstruction algorithm for HM . But in practice one takes a more direct approach.

Type Reconstruction for Hindley/Milner

Type reconstruction for the Hindley/Milner system works as for simply typed lambda calculus. We only have to add a clause for *let* expressions:

$TP : \text{Judgement} \rightarrow \text{Subst} \rightarrow \text{Subst}$

$TP(\Gamma \vdash t : T) s =$

case t **of**

...

let $x = t_1$ **in** t_2 : **let** a, b fresh **in**

let $s_1 = TP(\Gamma \vdash t_1 : a)$ **in**

$TP(\Gamma, x : \mathbf{gen}(s_1 \Gamma, s_1 a) \vdash t_2 : b) s_1$

where $\mathbf{gen}(\Gamma, T) = \forall tv(T) \setminus tv(\Gamma). T$.

