$$\text{type} \qquad (\textsc{nat})$$

$$\text{nat} \qquad (\textsc{z})$$

$$\frac{\text{nat}}{\text{nat}} \qquad (\textsc{s})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\\ \text{nat}\end{array}}{\text{type}} \qquad (\textsc{add})$$

$$\text{z} + \text{N} = \text{N} \qquad (\textsc{add}/\textsc{z})$$

$$\frac{\text{N1} + \text{N2} = \text{N3}}{\text{s(N1)} + \text{N2} = \text{s(N3)}} \qquad (\textsc{add}/\textsc{s})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\end{array}}{\text{type}} \qquad (\textsc{lte})$$

$$\text{z} \leq \text{N} \qquad (\textsc{lte}/\textsc{z})$$

$$\frac{\text{N1} \leq \text{N2}}{\text{s(N1)} \leq \text{s(N2)}} \qquad (\textsc{lte}/\textsc{s})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\end{array}}{\text{type}} \qquad (\textsc{eq})$$

$$\text{z} = \text{z} \qquad (\textsc{eq}/\textsc{z})$$

$$\frac{\text{N1} = \text{N2}}{\text{s(N1)} = \text{s(N2)}} \qquad (\textsc{eq}/\textsc{s})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\end{array}}{\text{type}} \qquad (\textsc{ne})$$

$$\text{z} \neq \text{s(N)} \qquad (\textsc{ne}/\textsc{z}1)$$

$$\text{s(N)} \neq \text{z} \qquad (\textsc{ne}/\textsc{z}2)$$

$$\frac{\text{N1} \neq \text{N2}}{\text{s(N1)} \neq \text{s(N2)}} \qquad (\textsc{ne}/\textsc{s})$$

$$\frac{\begin{array}{c}\text{N is nat}\\ \text{N} = \text{N}\end{array}}{\text{type}} \qquad (\textsc{eq-refl})$$

$$\text{eq-refl(z,eq/z)}$$

$$\frac{\text{eq-refl(N,E)}}{\text{eq-refl(s(N),eq/s(E))}}$$

$$\frac{\begin{array}{c}\text{A1 = A2}\\ \text{C1 = C2}\\ \text{A1 + B1 = C1}\\ \text{A2 + B2 = C2}\\ \text{B1 = B2}\end{array}}{\text{type}} \quad \text{(SUB-EQ)}$$

$$\text{sub-eq(eq/z,E,add/z,add/z,E)}$$

$$\frac{\text{sub-eq(E1,E2,A1,A2,E3)}}{\text{sub-eq(eq/s(E1),eq/s(E2),add/s(A1),add/s(A2),E3)}}$$

$$\frac{\begin{array}{c}\text{A + B = C}\\ \text{A + s(B) = s(C)}\end{array}}{\text{type}} \quad \text{(ADD-INC)}$$

$$\text{add-inc(add/z,add/z)}$$

$$\frac{\text{add-inc(A1,A2)}}{\text{add-inc(add/s(A1),add/s(A2))}}$$

$$\text{type} \quad \text{(EXP)}$$

$$\text{type} \quad \text{(TPE)}$$

$$\text{type} \quad \text{(VAL)}$$

$$\text{type} \quad \text{(VENV)}$$

$$\text{type} \quad \text{(TENV)}$$

$$\frac{\text{nat}}{\text{type}} \quad \text{(MEM)}$$

$$\text{mem(z)} \quad \text{(MNIL)}$$

$$\frac{\begin{array}{c}\text{N is nat}\\ \text{tpe}\\ \text{tpe}\\ \text{mem(N)}\end{array}}{\text{mem(s(N))}} \quad \text{(MCONS)}$$

$$\text{exp} \quad \text{(EMPTY)}$$

$$\frac{\text{nat}}{\text{exp}} \quad \text{(VAR)}$$

$$\frac{\begin{array}{c}\text{tpe}\\ \text{nat}\\ \text{tpe}\\ \text{exp}\\ \text{tpe}\\ \text{nat}\\ \text{exp}\\ \text{tpe}\\ \text{mem(N)}\end{array}}{\text{exp}} \quad \text{(FUN)}$$

$$\frac{\text{exp} \quad \text{nat} \quad \text{exp}}{\text{exp}} \quad (\textsc{app})$$

$$\frac{\text{exp} \quad \text{nat}}{\text{exp}} \quad (\textsc{sel})$$

$$\text{tpe} \quad (\textsc{top})$$

$$\text{tpe} \quad (\textsc{bot})$$

$$\frac{\text{nat} \quad \text{tpe} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{arrow})$$

$$\frac{\text{nat} \quad \text{tpe} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{rect})$$

$$\frac{\text{nat} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{recv})$$

$$\frac{\text{exp} \quad \text{nat}}{\text{tpe}} \quad (\textsc{tsel})$$

$$\frac{\text{nat} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{bind})$$

$$\frac{\text{tpe} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{and})$$

$$\frac{\text{tpe} \quad \text{tpe}}{\text{tpe}} \quad (\textsc{or})$$

$$\text{val} \quad (\textsc{emptyv})$$

$$\frac{\text{nat} \quad \text{exp} \quad \text{nat} \quad \text{val} \quad \text{venv}}{\text{val}} \quad (\textsc{clos})$$

$$\text{venv} \quad (\textsc{vnil})$$

$$\frac{\text{val} \quad \text{venv}}{\text{venv}} \qquad \text{(VCONS)}$$

$$\text{tenv} \qquad \text{(TNIL)}$$

$$\frac{\text{tpe} \quad \text{tenv}}{\text{tenv}} \qquad \text{(TCONS)}$$

$$\text{type} \qquad \text{(TOPT)}$$

$$\text{topt} \qquad \text{(TNONE)}$$

$$\frac{\text{tpe}}{\text{topt}} \qquad \text{(TSOME)}$$

$$\frac{\text{venv} \quad \text{nat} \quad \text{val}}{\text{type}} \qquad \text{(VLOOKUP-ZERO)}$$

$$\text{vlookup-zero(V::H,z,V)} \qquad \text{(VL/HIT)}$$

$$\frac{\text{vlookup-zero(H,N,V)}}{\text{vlookup-zero(V'::H,s(N),V)}} \qquad \text{(VL/MISS)}$$

$$\frac{\text{venv} \quad \text{nat}}{\text{type}} \qquad \text{(VSIZE)}$$

$$|\emptyset|=\text{z} \qquad \text{(VF/N)}$$

$$\frac{|\text{H}|=\text{N}}{|\text{V::H}|=\text{s(N)}} \qquad \text{(VF/C)}$$

$$\frac{\text{venv} \quad \text{nat} \quad \text{val}}{\text{type}} \qquad \text{(VLOOKUP)}$$

$$\frac{\text{vlookup-zero(G,M,V)} \quad \text{s(N)} + \text{M} = \text{S} \quad |\text{G}|=\text{S}}{\text{N} \mapsto \text{V} \in \text{G}} \qquad \text{(VL)}$$

$$\frac{\text{tenv} \quad \text{nat} \quad \text{tpe}}{\text{type}} \qquad \text{(TLOOKUP-ZERO)}$$

$$\text{tlookup-zero(V::G,z,V)} \qquad \text{(TL/HIT)}$$

$$\frac{\text{tlookup-zero(G,N,V)}}{\text{tlookup-zero(V'::G,s(N),V)}} \quad (\textsc{tl/miss})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{nat}\end{array}}{\text{type}} \quad (\textsc{tsize})$$

$$|\emptyset|=\text{z} \quad (\textsc{tf/n})$$

$$\frac{|\text{G}|=\text{N}}{|\text{V::G}|=\text{s(N)}} \quad (\textsc{tf/c})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{nat}\\ \text{tpe}\end{array}}{\text{type}} \quad (\textsc{tlookup})$$

$$\frac{\begin{array}{c}\text{tlookup-zero(G,M,V)}\\ \text{s(N)} + \text{M} = \text{S}\\ |\text{G}|=\text{S}\end{array}}{\text{N} \mapsto \text{V} \in \text{G}} \quad (\textsc{tl})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{tenv}\end{array}}{\text{type}} \quad (\textsc{sub-env})$$

$$\text{G} \sqsubseteq \text{G} \quad (\textsc{sub-env/refl})$$

$$\frac{\text{G1} \sqsubseteq \text{G2}}{\text{G1} \sqsubseteq \text{Z::G2}} \quad (\textsc{sub-env/ext})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{nat}\\ \text{tenv}\end{array}}{\text{type}} \quad (\textsc{sub-env-size})$$

$$\frac{\begin{array}{c}|\text{GN}|=\text{N}\\ \text{GN} \sqsubseteq \text{G}\end{array}}{\text{GN} \sqsubseteq_{\text{N}} \text{G}} \quad (\textsc{ses})$$

$$\frac{\begin{array}{c}\text{topt}\\ \text{topt}\\ \text{topt}\end{array}}{\text{type}} \quad (\textsc{topt-and})$$

$$\emptyset_T \wedge \emptyset_T = \emptyset_T \quad (\textsc{topt-and/nn})$$

$$\emptyset_T \wedge \text{T} = \text{T} \quad (\textsc{topt-and/ns})$$

$$\text{T} \wedge \emptyset_T = \text{T} \quad (\textsc{topt-and/sn})$$

$$\{\textbf{type}\ \ \text{N} : \text{TA1} \,..\, \text{TB1}\} \wedge \{\textbf{type}\ \ \text{N} : \text{TA2} \,..\, \text{TB2}\} = \{\textbf{type}\ \ \text{N} : \text{TA1} \vee \text{TA2} \,..\, \text{TB1} \wedge \text{TB2}\} \quad (\textsc{topt-and/ss})$$

$$\frac{\begin{array}{c}\text{topt}\\ \text{topt}\\ \text{topt}\end{array}}{\text{type}} \quad (\textsc{topt-or})$$

$$\emptyset_T \vee \emptyset_T = \emptyset_T \quad (\textsc{topt-or/nn})$$

$$\emptyset_T \vee \text{T} = \emptyset_T \quad (\textsc{topt-or/ns})$$

$$\text{T} \vee \emptyset_T = \emptyset_T \quad (\textsc{topt-or/sn})$$

$$\{\textbf{type}\ \ \text{N} : \text{TA1} \,..\, \text{TB1}\} \vee \{\textbf{type}\ \ \text{N} : \text{TA2} \,..\, \text{TB2}\} = \{\textbf{type}\ \ \text{N} : \text{TA1} \wedge \text{TA2} \,..\, \text{TB1} \vee \text{TB2}\} \quad (\textsc{topt-or/ss})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{tpe}\\ \text{nat}\\ \text{topt}\end{array}}{\text{type}} \quad (\textsc{exp-tp})$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{exp}\\ \text{tpe}\end{array}}{\text{type}} \quad (\textsc{tlookexp})$$

$$\text{G} \vdash \top \prec_{\text{N}} \emptyset_T \quad (\textsc{exp-tp/top})$$

$$\text{G} \vdash \bot \prec_{\text{N}} \emptyset_T \quad (\textsc{exp-tp/bot})$$

$$\text{G} \vdash \{\textbf{def}\ \ \text{N1} : \text{T1} \to \text{T2}\} \prec_{\text{N2}} \emptyset_T \quad (\textsc{exp-tp/fun})$$

$$\text{G} \vdash \{\textbf{val}\ \ \text{N1} : \text{T}\} \prec_{\text{N2}} \emptyset_T \quad (\textsc{exp-tp/recv})$$

$$\text{G} \vdash \{\textbf{type}\ \ \text{N} : \text{T1} \,..\, \text{T2}\} \prec_{\text{N}} \{\textbf{type}\ \ \text{N} : \text{T1} \,..\, \text{T2}\} \quad (\textsc{exp-tp/rect})$$

$$\frac{\text{N1} \neq \text{N2}}{\text{G} \vdash \{\textbf{type}\ \ \text{N1} : \text{T1} \,..\, \text{T2}\} \prec_{\text{N2}} \emptyset_T} \quad (\textsc{exp-tp/rectn})$$

$$\frac{\begin{array}{c}\text{T1'} \wedge \text{T2'} = \text{T3'}\\ \text{G} \vdash \text{T1} \prec_{\text{N}} \text{T1'}\\ \text{G} \vdash \text{T2} \prec_{\text{N}} \text{T2'}\end{array}}{\text{G} \vdash \text{T1} \wedge \text{T2} \prec_{\text{N}} \text{T3'}} \quad (\textsc{exp-tp/and})$$

$$\frac{\begin{array}{c}\text{T1'} \vee \text{T2'} = \text{T3'}\\ \text{G} \vdash \text{T1} \prec_{\text{N}} \text{T1'}\\ \text{G} \vdash \text{T2} \prec_{\text{N}} \text{T2'}\end{array}}{\text{G} \vdash \text{T1} \vee \text{T2} \prec_{\text{N}} \text{T3'}} \quad (\textsc{exp-tp/or})$$

$$\frac{\begin{array}{c}\{\ \text{N1} \Rightarrow \text{T1}\ \}::\text{G} \vdash \text{T1} \prec_{\text{N2}} \text{T2}\\ \text{G} \sqsubseteq_{\text{N1}} \text{G0}\end{array}}{\text{G0} \vdash \{\ \text{N1} \Rightarrow \text{T1}\ \} \prec_{\text{N2}} \text{T2}} \quad (\textsc{exp-tp/bind})$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{TB} \prec_{\text{N}} \text{T}\\ \text{G} \vdash \text{E} \ni_0 \{\textbf{type}\ \ \text{L} : \text{TA} \,..\, \text{TB}\}\end{array}}{\text{G} \vdash \text{E.L} \prec_{\text{N}} \text{T}} \quad (\textsc{exp-tp/tsel})$$

$$\frac{\mathrm{G} \vdash \mathrm{T0} \prec_{\mathrm{LN}} \{\mathbf{type}\ \ \mathrm{LN} : \mathrm{T1} \mathinner{..} \mathrm{T2}\} \quad \mathrm{N} \mapsto \mathrm{T0} \in \mathrm{G}}{\mathrm{G} \vdash \mathrm{var(N)} \ni_0 \{\mathbf{type}\ \ \mathrm{LN} : \mathrm{T1} \mathinner{..} \mathrm{T2}\}} \quad (\textsc{tle})$$

$$\mathrm{type} \quad (\textsc{mode})$$

$$\mathrm{mode} \quad (\textsc{strict})$$

$$\mathrm{mode} \quad (\textsc{lenient})$$

$$\frac{\mathrm{mode} \quad \mathrm{tenv} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{wf-tp})$$

$$\frac{\mathrm{mode} \quad \mathrm{tenv} \quad \mathrm{tpe} \quad \mathrm{tenv} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{sub-tp})$$

$$\frac{\mathrm{mode} \quad \mathrm{tenv} \quad \mathrm{exp} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{path-eval})$$

$$\frac{\mathrm{G} \vdash \mathrm{E} \ni_0 \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TS} \mathinner{..} \mathrm{TU}\}}{\mathrm{G} \vdash \mathrm{E} \ni_{\mathrm{M}} \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TS} \mathinner{..} \mathrm{TU}\}} \quad (\textsc{pev})$$

$$\frac{\mathrm{G} \vdash \mathrm{T}\ \mathbf{wf}_! \quad \mathrm{G} \sqsubseteq \mathrm{G2} \quad \mathrm{G} \sqsubseteq \mathrm{G1}}{\mathrm{G1} \vdash \mathrm{T} <:_! \mathrm{T} \dashv \mathrm{G2}} \quad (\textsc{sub-tp/refl})$$

$$\frac{\mathrm{G1} \vdash \mathrm{T}\ \mathbf{wf}_{\mathbf{M}}}{\mathrm{G1} \vdash \mathrm{T} <:_{\mathrm{M}} \top \dashv \mathrm{G2}} \quad (\textsc{sub-tp/top})$$

$$\frac{\mathrm{G1} \vdash \mathrm{TB1} <:_{\mathrm{M}} \mathrm{TB2} \dashv \mathrm{G2} \quad \mathrm{G2} \vdash \mathrm{TA2} <:_{\mathrm{M}} \mathrm{TA1} \dashv \mathrm{G1}}{\mathrm{G1} \vdash \{\mathbf{def}\ \ \mathrm{N} : \mathrm{TA1} \to \mathrm{TB1}\} <:_{\mathrm{M}} \{\mathbf{def}\ \ \mathrm{N} : \mathrm{TA2} \to \mathrm{TB2}\} \dashv \mathrm{G2}} \quad (\textsc{sub-tp/fun})$$

$$\frac{\mathrm{G1} \vdash \mathrm{TB1} <:_{\mathrm{M}} \mathrm{TB2} \dashv \mathrm{G2} \quad \mathrm{G2} \vdash \mathrm{TA2} <:_{\mathrm{M}} \mathrm{TA1} \dashv \mathrm{G1} \quad \mathrm{G2} \vdash \mathrm{TA2} <:_{\mathrm{M}} \mathrm{TB2} \dashv \mathrm{G2} \quad \mathrm{G1} \vdash \mathrm{TA1} <:_{\mathrm{M}} \mathrm{TB1} \dashv \mathrm{G1}}{\mathrm{G1} \vdash \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA1} \mathinner{..} \mathrm{TB1}\} <:_{\mathrm{M}} \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA2} \mathinner{..} \mathrm{TB2}\} \dashv \mathrm{G2}} \quad (\textsc{sub-tp/rect})$$

$$\frac{\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}{\mathrm{G1} \vdash \{\mathbf{val}\ \ \mathrm{N} : \mathrm{T1}\} <:_{\mathrm{M}} \{\mathbf{val}\ \ \mathrm{N} : \mathrm{T2}\} \dashv \mathrm{G2}} \quad (\textsc{sub-tp/recv})$$

$$\frac{\begin{array}{c}\mathrm{G1X} \vdash \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA1} \mathinner{..} \mathrm{TB1}\} <:_! \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA2} \mathinner{..} \mathrm{TB2}\} \dashv \mathrm{G2X} \\ \mathrm{G2X} \sqsubseteq_{\mathrm{s(X)}} \mathrm{G2} \\ \mathrm{G1X} \sqsubseteq_{\mathrm{s(X)}} \mathrm{G1} \\ \mathrm{G2} \vdash \mathrm{var(X)} \ni_! \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA2} \mathinner{..} \mathrm{TB2}\} \\ \mathrm{G1} \vdash \mathrm{var(X)} \ni_! \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA1} \mathinner{..} \mathrm{TB1}\}\end{array}}{\mathrm{G1} \vdash \mathrm{var(X).N} <:_! \mathrm{var(X).N} \dashv \mathrm{G2}}\ (\textsc{sub-tp/tselx-strict})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TN} \dashv \{\ \mathrm{N} \Rightarrow \mathrm{T1'}\ \}::\mathrm{G0} \\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T1'} \dashv \mathrm{G1'} \\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TN} \dashv \{\ \mathrm{N} \Rightarrow \mathrm{TN}\ \}::\mathrm{G0} \\ \mathrm{G0} \sqsubseteq_{\mathrm{N}} \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \{\ \mathrm{N} \Rightarrow \mathrm{TN}\ \} \dashv \mathrm{G2}}\ (\textsc{sub-tp/tbind2})$$

$$\frac{\begin{array}{c}\{\ \mathrm{N} \Rightarrow \mathrm{TN}\ \}::\mathrm{G0} \vdash \mathrm{TN} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2} \\ \mathrm{G0} \sqsubseteq_{\mathrm{N}} \mathrm{G1}\end{array}}{\mathrm{G1} \vdash \{\ \mathrm{N} \Rightarrow \mathrm{TN}\ \} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/tbind1})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TA2} \dashv \mathrm{G2} \\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TB2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TA2} \wedge \mathrm{TB2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/and2})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{TB1}\ \mathbf{wf_M} \\ \mathrm{G1} \vdash \mathrm{TA1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{TA1} \wedge \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/and1a})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{TA1}\ \mathbf{wf_M} \\ \mathrm{G1} \vdash \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{TA1} \wedge \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/and1b})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{TA1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2} \\ \mathrm{G1} \vdash \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{TA1} \vee \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/or2})$$

$$\frac{\begin{array}{c}\mathrm{G2} \vdash \mathrm{TB2}\ \mathbf{wf_M} \\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TA2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TA2} \vee \mathrm{TB2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/or1a})$$

$$\frac{\begin{array}{c}\mathrm{G2} \vdash \mathrm{TA2}\ \mathbf{wf_M} \\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TB2} \dashv \mathrm{G2}\end{array}}{\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{TA2} \vee \mathrm{TB2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/or1b})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{var(X).N}\ \mathbf{wf_M} \\ \mathrm{G1X} \vdash \mathrm{TB1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2} \\ \mathrm{G1X} \sqsubseteq_{\mathrm{s(X)}} \mathrm{G1} \\ \mathrm{G1} \vdash \mathrm{var(X)} \ni_{\mathrm{M}} \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA1} \mathinner{..} \mathrm{TB1}\}\end{array}}{\mathrm{G1} \vdash \mathrm{var(X).N} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}}\ (\textsc{sub-tp/tsel1})$$

$$\frac{\begin{array}{c}\mathrm{G2} \vdash \mathrm{var(X).N}\ \mathbf{wf_!} \\ \mathrm{G1} \vdash \mathrm{T1} <:_! \mathrm{TA2} \dashv \mathrm{G2X} \\ \mathrm{G2X} \sqsubseteq_{\mathrm{s(X)}} \mathrm{G2} \\ \mathrm{G2} \vdash \mathrm{var(X)} \ni_! \{\mathbf{type}\ \ \mathrm{N} : \mathrm{TA2} \mathinner{..} \mathrm{TB2}\}\end{array}}{\mathrm{G1} \vdash \mathrm{T1} <:_! \mathrm{var(X).N} \dashv \mathrm{G2}}\ (\textsc{sub-tp/tsel2-strict})$$

$$\frac{\begin{array}{c}\text{G2} \vdash \text{var(X).N } \mathbf{wf}_{\approx} \\ \text{G1} \vdash \text{T1} <:_{\approx} \text{TB2} \dashv \text{G2X} \\ \text{G2X} \sqsubseteq_{\text{s(X)}} \text{G2} \\ \text{G2} \vdash \text{var(X)} \ni_{\approx} \{\textbf{type } \text{N} : \text{TA2} .. \text{TB2}\}\end{array}}{\text{G1} \vdash \text{T1} <:_{\approx} \text{var(X).N} \dashv \text{G2}} \quad \text{(SUB-TP/TSEL2-LENIENT)}$$

$$\frac{\text{G2} \vdash \text{T } \mathbf{wf_M}}{\text{G1} \vdash \bot <:_{\text{M}} \text{T} \dashv \text{G2}} \quad \text{(SUB-TP/BOT)}$$

$$\text{G} \vdash \top \mathbf{wf_M} \quad \text{(WF-TP/TOP)}$$

$$\text{G} \vdash \bot \mathbf{wf_M} \quad \text{(WF-TP/BOT)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{TB } \mathbf{wf_M} \\ \text{G} \vdash \text{TA } \mathbf{wf_M}\end{array}}{\text{G} \vdash \{\textbf{def } \text{N} : \text{TA} \to \text{TB}\} \mathbf{wf_M}} \quad \text{(WF-TP/FUN)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{TA} <:_{\text{M}} \text{TB} \dashv \text{G} \\ \text{G} \vdash \text{TB } \mathbf{wf_M} \\ \text{G} \vdash \text{TA } \mathbf{wf_M}\end{array}}{\text{G} \vdash \{\textbf{type } \text{N} : \text{TA} .. \text{TB}\} \mathbf{wf_M}} \quad \text{(WF-TP/RECT)}$$

$$\frac{\text{G} \vdash \text{T } \mathbf{wf_M}}{\text{G} \vdash \{\textbf{val } \text{N} : \text{T}\} \mathbf{wf_M}} \quad \text{(WF-TP/RECV)}$$

$$\frac{\begin{array}{c}\text{GX} \vdash \{\textbf{type } \text{N} : \text{TA} .. \text{TB}\} \mathbf{wf_M} \\ \text{GX} \sqsubseteq_{\text{s(X)}} \text{G} \\ \text{G} \vdash \text{var(X)} \ni_{\text{M}} \{\textbf{type } \text{N} : \text{TA} .. \text{TB}\}\end{array}}{\text{G} \vdash \text{var(X).N } \mathbf{wf_M}} \quad \text{(WF-TP/TSEL)}$$

$$\frac{\begin{array}{c}\{\text{ N} \Rightarrow \text{TN }\}::\text{G0} \vdash \text{TN } \mathbf{wf_M} \\ \text{G0} \sqsubseteq_{\text{N}} \text{G}\end{array}}{\text{G} \vdash \{\text{ N} \Rightarrow \text{TN }\} \mathbf{wf_M}} \quad \text{(WF-TP/TBIND)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{TA } \mathbf{wf_M} \\ \text{G} \vdash \text{TB } \mathbf{wf_M}\end{array}}{\text{G} \vdash \text{TA} \wedge \text{TB } \mathbf{wf_M}} \quad \text{(WF-TP/AND)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{TA } \mathbf{wf_M} \\ \text{G} \vdash \text{TB } \mathbf{wf_M}\end{array}}{\text{G} \vdash \text{TA} \vee \text{TB } \mathbf{wf_M}} \quad \text{(WF-TP/OR)}$$

$$\frac{\begin{array}{c}\text{venv} \\ \text{exp} \\ \text{val}\end{array}}{\text{type}} \quad \text{(EVAL-EXP)}$$

$$\text{G} \vdash \{\emptyset_e\} \Downarrow \{\emptyset_v\} \quad \text{(E/EMPTY)}$$

$$\frac{\text{N} \mapsto \text{V} \in \text{G}}{\text{G} \vdash \text{var(N)} \Downarrow \text{V}} \quad \text{(E/VAR)}$$

$$\frac{< \{\textbf{def } \text{z} = \{\emptyset_e\}; \textbf{ val } \text{ z} = \{\emptyset_v\}\} \textbf{ in } \text{ G}>_v::\text{G} \vdash \text{R2} \Downarrow \text{V2}}{\text{G} \vdash \textbf{new } \text{TC } \{\textbf{def } \text{LNF}(\_:\text{X2}):\text{R}=\text{X3};\textbf{val } \text{LNV}:\text{R2}=\text{X4};\textbf{types } \text{MT}\} \Downarrow < \{\textbf{def } \text{LNF} = \text{R}; \textbf{ val } \text{ LNV} = \text{V2}\} \textbf{ in } \text{ G}>_v} \quad \text{(E/FUN)}$$

$$\frac{\mathrm{V2}::< \{\textbf{def}\ \mathrm{LNF} = \mathrm{R};\ \textbf{val}\ \ \mathrm{LNV} = \mathrm{R2}\}\ \textbf{in}\ \ \mathrm{G1}>_v::\mathrm{G1} \vdash \mathrm{R} \Downarrow \mathrm{V3} \quad \mathrm{G} \vdash \mathrm{E2} \Downarrow \mathrm{V2} \quad \mathrm{G} \vdash \mathrm{E1} \Downarrow < \{\textbf{def}\ \mathrm{LNF} = \mathrm{R};\ \textbf{val}\ \ \mathrm{LNV} = \mathrm{R2}\}\ \textbf{in}\ \ \mathrm{G1}>_v}{\mathrm{G} \vdash \mathrm{E1.LNF(E2)} \Downarrow \mathrm{V3}} \quad (\textsc{e/app})$$

$$\frac{\mathrm{G} \vdash \mathrm{E1} \Downarrow < \{\textbf{def}\ \mathrm{LNF} = \mathrm{R};\ \textbf{val}\ \ \mathrm{LNV} = \mathrm{V}\}\ \textbf{in}\ \ \mathrm{G1}>_v}{\mathrm{G} \vdash \mathrm{E1.LNV} \Downarrow \mathrm{V}} \quad (\textsc{e/sel})$$

$$\frac{\mathrm{mem(N)} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{type-mem})$$

$$\mathrm{type\text{-}mem(mnil,}\top\mathrm{)} \quad (\textsc{tm/nil})$$

$$\mathrm{type\text{-}mem(mcons(z,T1,T2,mnil),}\{\textbf{type}\ \ \mathrm{z} : \mathrm{T1} .. \mathrm{T2}\}\mathrm{)} \quad (\textsc{tm/consz})$$

$$\frac{\mathrm{type\text{-}mem(R,TR)}}{\mathrm{type\text{-}mem(mcons(s(N),T1,T2,R),}\{\textbf{type}\ \ \mathrm{s(N)} : \mathrm{T1} .. \mathrm{T2}\} \wedge \mathrm{TR}\mathrm{)}} \quad (\textsc{tm/cons})$$

$$\frac{\mathrm{tenv} \quad \mathrm{exp} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{type-exp})$$

$$\mathrm{G} \vdash \{\emptyset_e\} : \top \quad (\textsc{t/empty})$$

$$\frac{\mathrm{G} \vdash \mathrm{T}\ \mathbf{wf}_! \quad \mathrm{N} \mapsto \mathrm{T} \in \mathrm{G}}{\mathrm{G} \vdash \mathrm{var(N)} : \mathrm{T}} \quad (\textsc{t/var})$$

$$\frac{\mathrm{G} \vdash \mathrm{E1} : \{\textbf{val}\ \ \mathrm{LNV} : \mathrm{T1}\}}{\mathrm{G} \vdash \mathrm{E1.LNV} : \mathrm{T1}} \quad (\textsc{t/sel})$$

$$\frac{\begin{array}{c}\mathrm{TC::G} \vdash \{\textbf{def}\ \ \mathrm{LNF} : \mathrm{T3} \to \mathrm{T4}\} \wedge \{\textbf{val}\ \ \mathrm{LNV} : \mathrm{T2}\} \wedge \mathrm{MT} <:_! \mathrm{TC} \dashv \mathrm{TC::G} \\ \mathrm{G} \vdash \mathrm{TC}\ \mathbf{wf}_! \\ \mathrm{MT::G} \vdash \mathrm{MT}\ \mathbf{wf}_! \\ \mathrm{MT::G} \vdash \mathrm{T2} <:_! \mathrm{T2} \dashv \mathrm{TC::G} \\ \mathrm{MT::G} \vdash \mathrm{R2} : \mathrm{T2} \\ \mathrm{T3::TC::G} \vdash \mathrm{R} : \mathrm{T4} \\ \mathrm{type\text{-}mem(M,MT)} \\ |\mathrm{G}|=\mathrm{N}\end{array}}{\mathrm{G} \vdash \textbf{new}\ \mathrm{TC}\ \{\textbf{def}\ \mathrm{LNF}(\_:\mathrm{T3}):\mathrm{R}=\mathrm{T4};\textbf{val}\ \mathrm{LNV}:\mathrm{R2}=\mathrm{T2};\textbf{types}\ \mathrm{M}\} : \mathrm{TC}} \quad (\textsc{t/fun})$$

$$\frac{\mathrm{G} \vdash \mathrm{E2} : \mathrm{T1} \quad \mathrm{G} \vdash \mathrm{E1} : \{\textbf{def}\ \ \mathrm{LNF} : \mathrm{T1} \to \mathrm{T2}\}}{\mathrm{G} \vdash \mathrm{E1.LNF(E2)} : \mathrm{T2}} \quad (\textsc{t/app})$$

$$\frac{\mathrm{G} \vdash \mathrm{T1} <:_! \mathrm{T2} \dashv \mathrm{G} \quad \mathrm{G} \vdash \mathrm{E} : \mathrm{T1}}{\mathrm{G} \vdash \mathrm{E} : \mathrm{T2}} \quad (\textsc{t/sub})$$

$$\frac{\mathrm{val} \quad \mathrm{tenv} \quad \mathrm{tpe}}{\mathrm{type}} \quad (\textsc{wf-val})$$

$$\frac{\begin{array}{c}\text{venv}\\ \text{tenv}\end{array}}{\text{type}} \quad (\textsc{wf-env})$$

$$\{\emptyset_v\} \vdash \text{G } \mathbf{wf}_v \quad (\textsc{wfv/empty})$$

$$\frac{\begin{array}{c}\text{TC0::GC} \vdash \{\mathbf{def}\ \ \text{LNF} : \text{T1} \to \text{T2}\} \wedge \{\mathbf{val}\ \ \text{LNV} : \text{T}\} \wedge \text{TX} <:_{\approx} \text{TC0} \dashv \text{TC0::GC}\\ \text{TC0::GC} \vdash \text{TC0} <:_{\approx} \text{TC} \dashv \text{G}\\ \text{type-mem(M,TX)}\\ \text{R2} \vdash \text{TX::GC } \mathbf{wf}_v\\ \text{TX::GC} \vdash \text{T} <:_{\approx} \text{T} \dashv \text{TC0::GC}\\ \text{T1::TC0::GC} \vdash \text{R} : \text{T2}\\ \text{wf-env(H,GC)}\end{array}}{< \{\mathbf{def}\ \text{LNF} = \text{R};\ \mathbf{val}\ \ \text{LNV} = \text{R2}\}\ \mathbf{in}\ \ \text{H}>_v \vdash \text{G } \mathbf{wf}_v} \quad (\textsc{wfv/f})$$

$$\frac{\text{G} \vdash \{\mathbf{type}\ \ \text{LNT} : \text{T1} \mathinner{..} \text{T2}\}\ \mathbf{wf}_{\approx}}{< \{\mathbf{def}\ \text{LNF} = \text{R};\ \mathbf{val}\ \ \text{LNV} = \text{R2}\}\ \mathbf{in}\ \ \text{H}>_v \vdash \text{G } \mathbf{wf}_v} \quad (\textsc{wfv/t})$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\approx} \text{T} \dashv \text{G}\\ \text{V} \vdash \text{G1 } \mathbf{wf}_v\end{array}}{\text{V} \vdash \text{G } \mathbf{wf}_v} \quad (\textsc{wfv/sub})$$

$$\text{wf-env}(\emptyset,\emptyset) \quad (\textsc{wfe/n})$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{V} \vdash \text{T::G } \mathbf{wf}_v\end{array}}{\text{wf-env(V::H,T::G)}} \quad (\textsc{wfe/c})$$

$$\frac{\begin{array}{c}\text{G1} \sqsubseteq \text{G2}\\ \text{G2} \sqsubseteq \text{G3}\\ \text{G1} \sqsubseteq \text{G3}\end{array}}{\text{type}} \quad (\textsc{sub-env-trans})$$

$$\text{sub-env-trans(S,sub-env/refl,S)}$$

$$\frac{\text{sub-env-trans(S1,S2,S3)}}{\text{sub-env-trans(S1,sub-env/ext(S2),sub-env/ext(S3))}}$$

$$\frac{\begin{array}{c}\text{tenv}\\ \text{tenv}\\ \text{tenv}\\ \text{tenv}\end{array}}{\text{type}} \quad (\textsc{minmax-env})$$

$$\frac{\text{G1} \sqsubseteq \text{G2}}{\text{minmax-env(G1,G2,G1,G2)}} \quad (\textsc{mm-env}/1)$$

$$\frac{\text{G2} \sqsubseteq \text{G1}}{\text{minmax-env(G1,G2,G2,G1)}} \quad (\textsc{mm-env}/2)$$

$$\frac{\begin{array}{c}\text{G1} \sqsubseteq \text{GT}\\ \text{G2} \sqsubseteq \text{GT}\\ \text{minmax-env(G1,G2,GS,GU)}\end{array}}{\text{type}} \quad (\textsc{sub-env-dia})$$

$$\text{sub-env-dia(sub-env/refl,S,mm-env/2(S))}$$

$$\text{sub-env-dia(sub-env/ext(S1),sub-env/refl,mm-env/1(sub-env/ext(S1)))}$$

$$\frac{\text{sub-env-dia(S1,S2,MM)}}{\text{sub-env-dia(sub-env/ext(S1),sub-env/ext(S2),MM)}}$$

$$\frac{\begin{array}{c}\text{tlookup-zero(G,N,T)}\\ \text{Z is tpe}\\ \text{tlookup-zero(Z::G,s(N),T)}\end{array}}{\text{type}} \quad \text{(EXTEND-WF-LKPZ)}$$

$$\text{extend-wf-lkpz(tl/hit,X3,tl/miss(tl/hit))}$$

$$\frac{\text{extend-wf-lkpz(A,X4,B)}}{\text{extend-wf-lkpz(tl/miss(A),X5,tl/miss(B))}}$$

$$\frac{\begin{array}{c}\text{|G|=N}\\ \text{Z is tpe}\\ \text{|Z::G|=s(N)}\end{array}}{\text{type}} \quad \text{(SIZE-INC)}$$

$$\text{size-inc(tf/n,T,tf/c(tf/n))}$$

$$\frac{\text{size-inc(S,X3,S')}}{\text{size-inc(tf/c(S),T,tf/c(S'))}}$$

$$\frac{\begin{array}{c}\text{N} \mapsto \text{T} \in \text{G}\\ \text{Z is tpe}\\ \text{N} \mapsto \text{T} \in \text{Z::G}\end{array}}{\text{type}} \quad \text{(EXTEND-WF-LKP)}$$

$$\frac{\begin{array}{c}\text{extend-wf-lkpz(L,Z,L')}\\ \text{add-inc(A,A')}\\ \text{size-inc(S,Z,S')}\end{array}}{\text{extend-wf-lkp(tl(L,A,S),Z,tl(L',A',S'))}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{T} \prec_{\text{N}} \text{OT}\\ \text{Z is tpe}\\ \text{Z::G} \vdash \text{T} \prec_{\text{N}} \text{OT}\end{array}}{\text{type}} \quad \text{(EXTEND-EXP-TP)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{var(N)} \ni_0 \text{T}\\ \text{Z is tpe}\\ \text{Z::G} \vdash \text{var(N)} \ni_0 \text{T}\end{array}}{\text{type}} \quad \text{(EXTEND-WF-LKPE)}$$

$$\text{extend-exp-tp(exp-tp/top,Z,exp-tp/top)}$$

$$\text{extend-exp-tp(exp-tp/bot,Z,exp-tp/bot)}$$

$$\text{extend-exp-tp(exp-tp/fun,Z,exp-tp/fun)}$$

$$\text{extend-exp-tp(exp-tp/recv,Z,exp-tp/recv)}$$

extend-exp-tp(exp-tp/rect,Z,exp-tp/rect)

extend-exp-tp(exp-tp/rectn(N),Z,exp-tp/rectn(N))

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/and(topt-and/nn,E1,E2),Z,exp-tp/and(topt-and/nn,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/and(topt-and/ns,E1,E2),Z,exp-tp/and(topt-and/ns,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/and(topt-and/sn,E1,E2),Z,exp-tp/and(topt-and/sn,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/and(topt-and/ss,E1,E2),Z,exp-tp/and(topt-and/ss,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/or(topt-or/nn,E1,E2),Z,exp-tp/or(topt-or/nn,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/or(topt-or/ns,E1,E2),Z,exp-tp/or(topt-or/ns,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/or(topt-or/sn,E1,E2),Z,exp-tp/or(topt-or/sn,E1',E2'))}}$$

$$\frac{\text{extend-exp-tp(E2,Z,E2')} \quad \text{extend-exp-tp(E1,Z,E1')}}{\text{extend-exp-tp(exp-tp/or(topt-or/ss,E1,E2),Z,exp-tp/or(topt-or/ss,E1',E2'))}}$$

extend-exp-tp(exp-tp/bind(E,ses(N,S)),Z,exp-tp/bind(E,ses(N,sub-env/ext(S))))

$$\frac{\text{extend-exp-tp(E,Z,E')} \quad \text{extend-wf-lkpe(L,Z,L')}}{\text{extend-exp-tp(exp-tp/tsel(E,L),Z,exp-tp/tsel(E',L'))}}$$

$$\frac{\text{extend-exp-tp(E,Z,E')} \quad \text{extend-wf-lkp(L,Z,L')}}{\text{extend-wf-lkpe(tle(E,L),Z,tle(E',L'))}}$$

$$\frac{\mathrm{N} \mapsto \mathrm{T} \in \mathrm{G} \quad \mathrm{G} \sqsubseteq \mathrm{G1} \quad \mathrm{N} \mapsto \mathrm{T} \in \mathrm{G1}}{\text{type}} \qquad \text{(EXTEND-WF-LKP-MULT)}$$

extend-wf-lkp-mult(L,sub-env/refl,L)

$$\frac{\text{extend-wf-lkp(L1,X4,L2)} \quad \text{extend-wf-lkp-mult(L,S,L1)}}{\text{extend-wf-lkp-mult(L,sub-env/ext(S),L2)}}$$

$$\frac{\mathrm{G} \vdash \mathrm{T} \prec_{\mathrm{N}} \mathrm{OT} \quad \mathrm{G} \sqsubseteq \mathrm{G1} \quad \mathrm{G1} \vdash \mathrm{T} \prec_{\mathrm{N}} \mathrm{OT}}{\text{type}} \quad \text{(EXTEND-EXP-TP-MULT)}$$

$$\text{extend-exp-tp-mult(E,sub-env/refl,E)}$$

$$\frac{\text{extend-exp-tp(E1,X5,E2)} \quad \text{extend-exp-tp-mult(E,S,E1)}}{\text{extend-exp-tp-mult(E,sub-env/ext(S),E2)}}$$

$$\frac{\mathrm{G} \vdash \mathrm{var(N)} \ni_0 \mathrm{T} \quad \mathrm{G} \sqsubseteq \mathrm{G1} \quad \mathrm{G1} \vdash \mathrm{var(N)} \ni_0 \mathrm{T}}{\text{type}} \quad \text{(EXTEND-WF-LKPE-MULT)}$$

$$\text{extend-wf-lkpe-mult(L,sub-env/refl,L)}$$

$$\frac{\text{extend-wf-lkpe(L1,X4,L2)} \quad \text{extend-wf-lkpe-mult(L,S,L1)}}{\text{extend-wf-lkpe-mult(L,sub-env/ext(S),L2)}}$$

$$\frac{\text{tpe} \quad \text{tpe}}{\text{type}} \quad \text{(SAME)}$$

$$\text{same(T,T)} \quad \text{(IDENT)}$$

$$\frac{\text{topt} \quad \text{topt}}{\text{type}} \quad \text{(SAMEOPT)}$$

$$\text{sameopt(T,T)} \quad \text{(IDENTOPT)}$$

$$\frac{\text{tenv} \quad \text{tenv}}{\text{type}} \quad \text{(SAMETENV)}$$

$$\text{sametenv(G,G)} \quad \text{(IDENTTENV)}$$

$$\text{type} \quad \text{(FALSE)}$$

$$\frac{\mathrm{N} \neq \mathrm{N} \quad \text{false}}{\text{type}} \quad \text{(NE-IRREFL)}$$

$$\frac{\text{ne-irrefl(NE,F)}}{\text{ne-irrefl(ne/s(NE),F)}}$$

$$\frac{\begin{array}{c}\text{false}\\ \text{T1 is tpe}\\ \text{T2 is tpe}\\ \text{same(T1,T2)}\end{array}}{\text{type}}\ (\textsc{no-eq})$$

$$\frac{\begin{array}{c}\text{false}\\ \text{T1 is topt}\\ \text{T2 is topt}\\ \text{sameopt(T1,T2)}\end{array}}{\text{type}}\ (\textsc{no-eq2})$$

$$\frac{\begin{array}{c}\text{false}\\ \text{T1 is tenv}\\ \text{T2 is tenv}\\ \text{sametenv(T1,T2)}\end{array}}{\text{type}}\ (\textsc{no-eq-env})$$

$$\frac{\begin{array}{c}\text{false}\\ \text{M is mode}\\ \text{G is tenv}\\ \text{E is exp}\\ \text{T is tpe}\\ \text{G} \vdash \text{E} \ni_{\text{M}} \text{T}\end{array}}{\text{type}}\ (\textsc{no-pev})$$

$$\frac{\begin{array}{c}\text{same(T1,T1')}\\ \text{G} \vdash \text{T1} \prec_{\text{N}} \text{T2}\\ \text{G} \vdash \text{T1'} \prec_{\text{N}} \text{T2}\end{array}}{\text{type}}\ (\textsc{eq-exp-low})$$

$$\text{eq-exp-low(ident,S,S)}$$

$$\frac{\begin{array}{c}\text{same(T1,T1')}\\ \text{same(T2,T2')}\\ \text{LN is nat}\\ \text{same}(\{\textbf{type}\ \ \text{LN} : \text{T1} \mathinner{..} \text{T2}\},\{\textbf{type}\ \ \text{LN} : \text{T1'} \mathinner{..} \text{T2'}\})\\ \text{same}(\{\textbf{type}\ \ \text{LN} : \text{T1'} \mathinner{..} \text{T2'}\},\{\textbf{type}\ \ \text{LN} : \text{T1} \mathinner{..} \text{T2}\})\end{array}}{\text{type}}\ (\textsc{eq-rect})$$

$$\text{eq-rect(ident,ident,X3,ident,ident)}$$

$$\frac{\begin{array}{c}\text{sameopt}(\{\textbf{type}\ \ \text{LN} : \text{T1} \mathinner{..} \text{T2}\},\{\textbf{type}\ \ \text{LN} : \text{T1'} \mathinner{..} \text{T2'}\})\\ \text{same(T1,T1')}\\ \text{same(T2,T2')}\end{array}}{\text{type}}\ (\textsc{eq-rect2})$$

$$\text{eq-rect2(identopt,ident,ident)}$$

$$\frac{\begin{array}{c}\text{same(T1,T1')}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \text{T1'} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}}\ (\textsc{eq-low})$$

$$\text{eq-low(ident,S,S)}$$

$$\frac{\begin{array}{c}\text{sametenv(G1,G1')}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1'} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(EQ-LOW-ENV)}$$

eq-low-env(identtenv,S,S)

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\end{array}}{\text{type}} \quad \text{(NATID)}$$

$$\text{natid(N,N)} \quad \text{(NATIDENT)}$$

$$\frac{\begin{array}{c}\text{natid(N1,N2)}\\ \text{natid(s(N1),s(N2))}\end{array}}{\text{type}} \quad \text{(NATID-SUCC)}$$

natid-succ(natident,natident)

$$\frac{\begin{array}{c}\text{N1 = N2}\\ \text{natid(N1,N2)}\end{array}}{\text{type}} \quad \text{(EQ-TO-ID)}$$

eq-to-id(eq/z,natident)

$$\frac{\begin{array}{c}\text{natid-succ(B,B')}\\ \text{eq-to-id(A,B)}\end{array}}{\text{eq-to-id(eq/s(A),B')}}$$

$$\frac{\begin{array}{c}\text{C = C'}\\ \text{C' = C}\end{array}}{\text{type}} \quad \text{(EQ-SYM)}$$

eq-sym(eq/z,eq/z)

$$\frac{\text{eq-sym(A,B)}}{\text{eq-sym(eq/s(A),eq/s(B))}}$$

$$\frac{\begin{array}{c}\text{natid(N1,N1')}\\ \text{G1} \vdash \{\textbf{val}\ \ \text{N1} : \text{T1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \{\textbf{val}\ \ \text{N1'} : \text{T1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(NATID-RECV-LOW)}$$

natid-recv-low(natident,S,S)

$$\frac{\begin{array}{c}\text{N1 = N1'}\\ \text{G1} \vdash \{\textbf{val}\ \ \text{N1} : \text{T1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \{\textbf{val}\ \ \text{N1'} : \text{T1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(EQ-RECV-LOW)}$$

$$\frac{\begin{array}{c}\text{natid-recv-low(ID,S,S')}\\ \text{eq-to-id(EQ,ID)}\end{array}}{\text{eq-recv-low(EQ,S,S')}}$$

$$\frac{\begin{array}{c}\text{natid(N1,N1')}\\ \text{G1} \vdash \{\textbf{def}\ \ \text{N1} : \text{TA1} \to \text{TB1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \{\textbf{def}\ \ \text{N1'} : \text{TA1} \to \text{TB1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(NATID-ARROW-LOW)}$$

$$\text{natid-arrow-low(natident,S,S)}$$

$$\frac{\begin{array}{c}\text{N1 = N1'}\\ \text{G1} \vdash \{\textbf{def}\ \ \text{N1} : \text{TA1} \to \text{TB1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \{\textbf{def}\ \ \text{N1'} : \text{TA1} \to \text{TB1}\} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(EQ-ARROW-LOW)}$$

$$\frac{\begin{array}{c}\text{natid-arrow-low(ID,S,S')}\\ \text{eq-to-id(EQ,ID)}\end{array}}{\text{eq-arrow-low(EQ,S,S')}}$$

$$\frac{\begin{array}{c}\text{natid(LNF,LNF')}\\ < \{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G}\ \mathbf{wf}_v\\ < \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G}\ \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(NATID-EQ-WFV-CLOS)}$$

$$\text{natid-eq-wfv-clos(natident,S,S)}$$

$$\frac{\begin{array}{c}\text{LNF = LNF'}\\ < \{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G}\ \mathbf{wf}_v\\ < \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G}\ \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(EQ-WFV-CLOS)}$$

$$\frac{\begin{array}{c}\text{natid-eq-wfv-clos(ID,S,S')}\\ \text{eq-to-id(EQ,ID)}\end{array}}{\text{eq-wfv-clos(EQ,S,S')}}$$

$$\frac{\begin{array}{c}\text{same(T2,T2')}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2'} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(EQ-HIGH)}$$

$$\text{eq-high(ident,S,S)}$$

$$\frac{\begin{array}{c}\text{sametenv(G2,G2')}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2'}\end{array}}{\text{type}} \quad \text{(EQ-HIGH-ENV)}$$

$$\text{eq-high-env(identtenv,S,S)}$$

$$\frac{\begin{array}{c}\text{same(T1,T1')}\\ \text{G1} \vdash \text{T1}\ \mathbf{wf_M}\\ \text{G1} \vdash \text{T1'}\ \mathbf{wf_M}\end{array}}{\text{type}} \quad \text{(EQ-WF)}$$

$$\text{eq-wf(ident,S,S)}$$

$$\frac{\begin{array}{c}\text{sametenv(G1,G1')}\\ \text{G1} \vdash \text{T1 } \mathbf{wf_M}\\ \text{G1'} \vdash \text{T1 } \mathbf{wf_M}\end{array}}{\text{type}} \quad \text{(EQ-WF-ENV)}$$

$$\text{eq-wf-env(identtenv,S,S)}$$

$$\frac{\begin{array}{c}\text{same(T1,T1')}\\ \text{G1} \vdash \text{N} \ni_{\text{M}} \text{T1}\\ \text{G1} \vdash \text{N} \ni_{\text{M}} \text{T1'}\end{array}}{\text{type}} \quad \text{(EQ-PEV)}$$

$$\text{eq-pev(ident,S,S)}$$

$$\frac{\begin{array}{c}\text{sametenv(G,G')}\\ \text{G} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G'} \vdash \text{N} \ni_{\text{M}} \text{T}\end{array}}{\text{type}} \quad \text{(EQ-PEV-ENV)}$$

$$\text{eq-pev-env(identtenv,S,S)}$$

$$\frac{\begin{array}{c}\text{tlookup-zero(G2,E,T1)}\\ \text{tlookup-zero(G2,E,T2)}\\ \text{same(T1,T2)}\end{array}}{\text{type}} \quad \text{(LOOKUP-ZERO-SAME)}$$

$$\text{lookup-zero-same(tl/hit,tl/hit,ident)}$$

$$\frac{\text{lookup-zero-same(A,B,X)}}{\text{lookup-zero-same(tl/miss(A),tl/miss(B),X)}}$$

$$\frac{\begin{array}{c}|\text{G2}|=\text{N1}\\ |\text{G2}|=\text{N2}\\ \text{N1} = \text{N2}\end{array}}{\text{type}} \quad \text{(SIZE-EQ)}$$

$$\text{size-eq(tf/n,tf/n,eq/z)}$$

$$\frac{\text{size-eq(A,B,X)}}{\text{size-eq(tf/c(A),tf/c(B),eq/s(X))}}$$

$$\frac{\begin{array}{c}\text{tlookup-zero(G2,N1,\{}\mathbf{type}\ \ \text{LN : T1 .. T3\})}\\ \text{tlookup-zero(G2,N2,\{}\mathbf{type}\ \ \text{LN : T2 .. T4\})}\\ \text{N1 = N2}\\ \text{same(T1,T2)}\\ \text{same(T3,T4)}\end{array}}{\text{type}} \quad \text{(LOOKUP-ZERO-EQ)}$$

$$\text{lookup-zero-eq(tl/hit,tl/hit,eq/z,ident,ident)}$$

$$\frac{\text{lookup-zero-eq(A,B,E,X,Y)}}{\text{lookup-zero-eq(tl/miss(A),tl/miss(B),eq/s(E),X,Y)}}$$

$$\frac{\begin{array}{c}N1 \mapsto \{\textbf{type}\ \ LN : T1\ ..\ T3\} \in G2 \\ N2 \mapsto \{\textbf{type}\ \ LN : T2\ ..\ T4\} \in G2 \\ N1 = N2 \\ \text{same}(T1,T2) \\ \text{same}(T3,T4)\end{array}}{\text{type}} \quad \text{(LOOKUP-EQ)}$$

$$\frac{\begin{array}{c}\text{lookup-zero-eq}(L1,L2,EM,X,Y) \\ \text{sub-eq}(EN,ES,A1,A2,EM) \\ \text{size-eq}(F1,F2,ES)\end{array}}{\text{lookup-eq}(\text{tl}(L1,\text{add/s}(A1),\text{tf/c}(F1)),\text{tl}(L2,\text{add/s}(A2),\text{tf/c}(F2)),EN,X,Y)}$$

$$\frac{\begin{array}{c}\text{tlookup-zero}(G2,N1,T1) \\ \text{tlookup-zero}(G2,N2,T2) \\ N1 = N2 \\ \text{same}(T1,T2)\end{array}}{\text{type}} \quad \text{(LOOKUP-ZERO-EQ0)}$$

$$\text{lookup-zero-eq0}(\text{tl/hit},\text{tl/hit},\text{eq/z},\text{ident})$$

$$\frac{\text{lookup-zero-eq0}(A,B,E,X)}{\text{lookup-zero-eq0}(\text{tl/miss}(A),\text{tl/miss}(B),\text{eq/s}(E),X)}$$

$$\frac{\begin{array}{c}N1 \mapsto T1 \in G2 \\ N2 \mapsto T2 \in G2 \\ N1 = N2 \\ \text{same}(T1,T2)\end{array}}{\text{type}} \quad \text{(LOOKUP-EQ0)}$$

$$\frac{\begin{array}{c}\text{lookup-zero-eq0}(L1,L2,EM,X) \\ \text{sub-eq}(EN,ES,A1,A2,EM) \\ \text{size-eq}(F1,F2,ES)\end{array}}{\text{lookup-eq0}(\text{tl}(L1,\text{add/s}(A1),\text{tf/c}(F1)),\text{tl}(L2,\text{add/s}(A2),\text{tf/c}(F2)),EN,X)}$$

$$\frac{\begin{array}{c}\text{same}(T,T') \\ N \mapsto T \in G \\ N \mapsto T' \in G\end{array}}{\text{type}} \quad \text{(EQ-LOOKUP)}$$

$$\text{eq-lookup}(\text{ident},L,L)$$

$$\frac{\begin{array}{c}T1 \wedge T2 = T3 \\ T1 \wedge T2 = T3' \\ \text{sameopt}(T3,T3')\end{array}}{\text{type}} \quad \text{(TOPT-AND-EQ)}$$

$$\text{topt-and-eq}(\text{topt-and/nn},\text{topt-and/nn},\text{identopt})$$

$$\text{topt-and-eq}(\text{topt-and/ns},\text{topt-and/ns},\text{identopt})$$

$$\text{topt-and-eq}(\text{topt-and/sn},\text{topt-and/sn},\text{identopt})$$

$$\text{topt-and-eq}(\text{topt-and/ss},\text{topt-and/ss},\text{identopt})$$

$$\frac{\text{T1} \vee \text{T2} = \text{T3} \quad \text{T1} \vee \text{T2} = \text{T3'} \quad \text{sameopt(T3,T3')}}{\text{type}} \quad \text{(TOPT-OR-EQ)}$$

topt-or-eq(topt-or/nn,topt-or/nn,identopt)

topt-or-eq(topt-or/ns,topt-or/ns,identopt)

topt-or-eq(topt-or/sn,topt-or/sn,identopt)

topt-or-eq(topt-or/ss,topt-or/ss,identopt)

$$\frac{\text{sameopt(T1,T1')} \quad \text{sameopt(T2,T2')} \quad \text{T1} \wedge \text{T2} = \text{T3} \quad \text{T1'} \wedge \text{T2'} = \text{T3}}{\text{type}} \quad \text{(EQ-TOPT-AND)}$$

eq-topt-and(identopt,identopt,T,T)

$$\frac{\text{sameopt(T1,T1')} \quad \text{sameopt(T2,T2')} \quad \text{T1} \vee \text{T2} = \text{T3} \quad \text{T1'} \vee \text{T2'} = \text{T3}}{\text{type}} \quad \text{(EQ-TOPT-OR)}$$

eq-topt-or(identopt,identopt,T,T)

$$\frac{\text{T::GN} \sqsubseteq \text{G} \quad \text{GN} \sqsubseteq \text{G}}{\text{type}} \quad \text{(SUB-ENV-CONS)}$$

sub-env-cons(sub-env/refl,sub-env/ext(sub-env/refl))

$$\frac{\text{sub-env-cons(S,S')}}{\text{sub-env-cons(sub-env/ext(S),sub-env/ext(S'))}}$$

$$\frac{\text{sametenv(T1::G1,T2::G2)} \quad \text{sametenv(G1,G2)}}{\text{type}} \quad \text{(TCONS-SAMETENV-EQ)}$$

tcons-sametenv-eq(identtenv,identtenv)

$$\frac{\text{sametenv(G,G')} \quad \text{T is tpe} \quad \text{sametenv(T::G,T::G')}}{\text{type}} \quad \text{(EQ-SAMETENV-CONS)}$$

eq-sametenv-cons(identtenv,X2,identtenv)

$$\frac{\text{sametenv(G,G')} \quad \text{same(T,T')} \quad \text{sametenv(T::G,T'::G')}}{\text{type}} \quad \text{(EQ2-SAMETENV-CONS)}$$

eq2-sametenv-cons(identtenv,ident,identtenv)

$$\frac{\begin{array}{c}\text{false}\\ \text{G is tenv}\\ \text{G' is tenv}\\ \text{G} \sqsubseteq \text{G'}\end{array}}{\text{type}} \quad \text{(NO-SUB-ENV)}$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\end{array}}{\text{type}} \quad \text{(LT)}$$

$$\text{lt(z,s(N))} \quad \text{(LT/Z)}$$

$$\frac{\text{lt(N1,N2)}}{\text{lt(s(N1),s(N2))}} \quad \text{(LT/S)}$$

$$\frac{\begin{array}{c}\text{lt(N,z)}\\ \text{false}\end{array}}{\text{type}} \quad \text{(LT-Z-FALSE)}$$

$$\frac{\begin{array}{c}\text{s(N)} \leq \text{z}\\ \text{false}\end{array}}{\text{type}} \quad \text{(LTE-Z-FALSE)}$$

$$\frac{\begin{array}{c}\text{s(N)} + \text{M} = \text{S}\\ \text{lt(N,S)}\end{array}}{\text{type}} \quad \text{(ADD-LT)}$$

$$\text{add-lt(add/s(add/z),lt/z)}$$

$$\frac{\text{add-lt(A,LT)}}{\text{add-lt(add/s(A),lt/s(LT))}}$$

$$\frac{\begin{array}{c}\text{N0} \mapsto \text{T} \in \text{G}\\ |\text{G}|=\text{N}\\ \text{lt(N0,N)}\end{array}}{\text{type}} \quad \text{(TSIZE-LOOKUP)}$$

$$\frac{\text{add-lt(A,LT)}}{\text{tsize-lookup(tl(L,A,N),N,LT)}}$$

$$\frac{\begin{array}{c}\text{A} \leq \text{B}\\ \text{A} \leq \text{s(B)}\end{array}}{\text{type}} \quad \text{(LTE-INC)}$$

$$\text{lte-inc(lte/z,lte/z)}$$

$$\frac{\text{lte-inc(A,B)}}{\text{lte-inc(lte/s(A),lte/s(B))}}$$

$$\frac{\begin{array}{c}\text{GA} \sqsubseteq \text{GB}\\ |\text{GA}|=\text{NA}\\ |\text{GB}|=\text{NB}\\ \text{NA} \leq \text{NB}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-CALC)}$$

sub-env-size-calc(S,tf/n,SB,lte/z)

$$\frac{\text{sub-env-size-calc(sub-env/refl,SA,SB,LT)}}{\text{sub-env-size-calc(sub-env/refl,tf/c(SA),tf/c(SB),lte/s(LT))}}$$

$$\frac{\begin{array}{c}\text{lte-inc(LT,LT')}\\ \text{sub-env-size-calc(S,tf/c(SA),SB,LT)}\end{array}}{\text{sub-env-size-calc(sub-env/ext(S),tf/c(SA),tf/c(SB),LT')}}$$

$$\frac{\begin{array}{c}\text{NA} = \text{NB}\\ \text{NA} \leq \text{NB}\end{array}}{\text{type}} \quad \text{(EQ-LTE)}$$

eq-lte(eq/z,lte/z)

$$\frac{\text{eq-lte(A,B)}}{\text{eq-lte(eq/s(A),lte/s(B))}}$$

$$\frac{\begin{array}{c}\text{NA} \leq \text{NB}\\ \text{NB} \leq \text{NA}\\ \text{NA} = \text{NB}\end{array}}{\text{type}} \quad \text{(LTE-LTE-EQ)}$$

lte-lte-eq(lte/z,lte/z,eq/z)

$$\frac{\text{lte-lte-eq(A,B,C)}}{\text{lte-lte-eq(lte/s(A),lte/s(B),eq/s(C))}}$$

$$\frac{\begin{array}{c}\text{z} = \text{s(z)}\\ \text{false}\end{array}}{\text{type}} \quad \text{(EQ-Z-SZ-CONTRA)}$$

$$\frac{\begin{array}{c}\text{N1} = \text{N2}\\ \text{N1} = \text{s(N2)}\\ \text{false}\end{array}}{\text{type}} \quad \text{(EQ-EQ-S-CONTRA)}$$

$$\frac{\text{eq-z-sz-contra(EQ,CONTRA)}}{\text{eq-eq-s-contra(eq/z,EQ,CONTRA)}}$$

$$\frac{\text{eq-eq-s-contra(A,B,C)}}{\text{eq-eq-s-contra(eq/s(A),eq/s(B),C)}}$$

$$\frac{\begin{array}{c}\text{s(X1)} \leq \text{X2}\\ \text{X2} \leq \text{s(X1)}\\ \text{s(X1)} \leq \text{s(X2)}\\ \text{X2} \leq \text{X1}\\ \text{false}\end{array}}{\text{type}} \quad \text{(LTE-CONTRA)}$$

$$\frac{\begin{array}{c}\text{eq-eq-s-contra(EQA,EQB,CONTRA)}\\ \text{lte-lte-eq(A1,A2,EQA)}\\ \text{lte-lte-eq(B1,B2,EQB)}\end{array}}{\text{lte-contra(B2,B1,lte/s(A2),A1,CONTRA)}}$$

$$\frac{\begin{array}{c}\mathrm{GA} \sqsubseteq \mathrm{GB}\\ |\mathrm{GA}|{=}\mathrm{NA}\\ |\mathrm{GB}|{=}\mathrm{NB}\\ \mathrm{NB} \leq \mathrm{NA}\\ \mathrm{NA} \leq \mathrm{NB}\\ \mathrm{GB} \sqsubseteq \mathrm{GA}\end{array}}{\text{type}} \quad (\textsc{sub-env-size0})$$

$$\frac{\begin{array}{c}\text{eq-lte(EQ,LT)}\\ \text{eq-refl(X1,EQ)}\end{array}}{\text{sub-env-size0(sub-env/refl,X3,X4,X5,LT,sub-env/refl)}}$$

$$\frac{\text{sub-env-size0(sub-env/refl,A,B,LT,LT',X4)}}{\text{sub-env-size0(sub-env/refl,tf/c(A),tf/c(B),lte/s(LT),lte/s(LT'),sub-env/refl)}}$$

$$\frac{\begin{array}{c}\text{no-sub-env(CONTRA,X1::X2,}\emptyset\text{,OUT)}\\ \text{lte-z-false(LT,CONTRA)}\end{array}}{\text{sub-env-size0(sub-env/ext(S),tf/n,tf/c(B),LT,lte/z,OUT)}}$$

$$\frac{\begin{array}{c}\text{no-sub-env(CONTRA,X1::X2,X3::X4,OUT)}\\ \text{lte-contra(LT2,LT',LT2',LT,CONTRA)}\\ \text{lte-inc(LT2,LT2')}\\ \text{sub-env-size0(S,tf/c(A),B,LT',LT2,X7)}\\ \text{lte-inc(LT,LT')}\end{array}}{\text{sub-env-size0(sub-env/ext(S),tf/c(A),tf/c(B),lte/s(LT),LT2',OUT)}}$$

$$\frac{\begin{array}{c}\text{lt(NB,NA)}\\ \mathrm{s(NB)} \leq \mathrm{NA}\end{array}}{\text{type}} \quad (\textsc{lt-lte})$$

$$\text{lt-lte(lt/z,lte/s(lte/z))}$$

$$\frac{\text{lt-lte(A,B)}}{\text{lt-lte(lt/s(A),lte/s(B))}}$$

$$\frac{\begin{array}{c}\mathrm{GA} \sqsubseteq \mathrm{T0{::}GB}\\ |\mathrm{GA}|{=}\mathrm{NA}\\ |\mathrm{GB}|{=}\mathrm{NB}\\ \text{lt(NB,NA)}\\ \mathrm{T0{::}GB} \sqsubseteq \mathrm{GA}\end{array}}{\text{type}} \quad (\textsc{sub-env-size1})$$

$$\frac{\begin{array}{c}\text{sub-env-size0(S,SNA,tf/c(SNB),LTE,X6,OUT)}\\ \text{lt-lte(LT,LTE)}\end{array}}{\text{sub-env-size1(S,SNA,SNB,LT,OUT)}}$$

$$\frac{\begin{array}{c}\mathrm{G} \sqsubseteq \mathrm{G2}\\ \mathrm{T0{::}G0} \sqsubseteq \mathrm{G2}\\ |\mathrm{G}|{=}\mathrm{N}\\ |\mathrm{G0}|{=}\mathrm{N0}\\ \text{lt(N0,N)}\\ \mathrm{T0{::}G0} \sqsubseteq \mathrm{G}\end{array}}{\text{type}} \quad (\textsc{sub-env-size0b})$$

$$\frac{}{\text{sub-env-size0b(sub-env/refl,S,X6,X7,X8,S)}}$$

$$\frac{\text{sub-env-size0b(S,S0,SN,SN0,LT,OUT)}}{\text{sub-env-size0b(sub-env/ext(S),sub-env/ext(S0),SN,SN0,LT,OUT)}}$$

$$\frac{\text{sub-env-size1(sub-env/ext(S),SN,SN0,LT,OUT)}}{\text{sub-env-size0b(sub-env/ext(S),sub-env/refl,SN,SN0,LT,OUT)}}$$

$$\frac{\begin{array}{c}\text{|G0|=N0}\\ \text{G} \sqsubseteq \text{G1}\\ \text{G} \sqsubseteq \text{G2}\\ \text{T0::G0} \sqsubseteq \text{G2}\\ \text{G} \vdash \text{var(N0)} \ni_{\text{M}} \{\textbf{type}\ \ \text{N} : \text{TA} \mathinner{..} \text{TB}\}\\ \text{T0::G0} \sqsubseteq \text{G}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-LKP)}$$

$$\frac{\begin{array}{c}\text{sub-env-size0b(S2,S0,SN,SN0,LT,OUT)}\\ \text{tsize-lookup(L,SN,LT)}\end{array}}{\text{sub-env-size-lkp(SN0,S1,S2,S0,pev(tle(EX,L)),OUT)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{var(X)} \ni_{\text{M}} \text{T}\\ \text{X} \mapsto \text{TX} \in \text{G}\end{array}}{\text{type}} \quad \text{(EXTRACT-PEV-LKP)}$$

$$\frac{}{\text{extract-pev-lkp(pev(tle(EXR,LR)),LR)}}$$

$$\frac{\begin{array}{c}\text{GN} \sqsubseteq \text{G}\\ \text{|GN|=NN}\\ \text{|G|=N}\\ \text{NN} \leq \text{N}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-LTE)}$$

$$\frac{\begin{array}{c}\text{eq-lte(EQ,LT)}\\ \text{size-eq(A,B,EQ)}\end{array}}{\text{sub-env-size-lte(sub-env/refl,A,B,LT)}}$$

$$\frac{\begin{array}{c}\text{lte-inc(LT,LT')}\\ \text{sub-env-size-lte(S,A,B,LT)}\end{array}}{\text{sub-env-size-lte(sub-env/ext(S),A,tf/c(B),LT')}}$$

$$\frac{\begin{array}{c}\text{s(N)} \leq \text{N}\\ \text{false}\end{array}}{\text{type}} \quad \text{(LTE-S-FALSE)}$$

$$\frac{\text{lte-s-false(A,CONTRA)}}{\text{lte-s-false(lte/s(A),CONTRA)}}$$

$$\frac{\begin{array}{c}\text{sametenv(G1,G2)}\\ \text{T1::G1} \sqsubseteq \text{G}\\ \text{T2::G2} \sqsubseteq \text{G}\\ \text{|G1|=N}\\ \text{|G2|=N}\\ \text{same(T1,T2)}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-EQ-CONS)}$$

$$\text{sub-env-size-eq-cons(GEQ,sub-env/refl,sub-env/refl,N1,N2,ident)}$$

$$\frac{\text{sub-env-size-eq-cons(GEQ,A1,A2,N1,N2,TEQ)}}{\text{sub-env-size-eq-cons(GEQ,sub-env/ext(A1),sub-env/ext(A2),N1,N2,TEQ)}}$$

$$\frac{\begin{array}{c}\text{no-eq(CONTRA,X1,X2,TEQ)}\\ \text{lte-s-false(LT,CONTRA)}\\ \text{sub-env-size-lte(A2,tf/c(N2),N1,LT)}\end{array}}{\text{sub-env-size-eq-cons(GEQ,sub-env/refl,sub-env/ext(A2),N1,N2,TEQ)}}$$

$$\frac{\begin{array}{c}\text{no-eq(CONTRA,X1,X2,TEQ)}\\ \text{lte-s-false(LT,CONTRA)}\\ \text{sub-env-size-lte(A1,tf/c(N1),N2,LT)}\end{array}}{\text{sub-env-size-eq-cons(GEQ,sub-env/ext(A1),sub-env/refl,N1,N2,TEQ)}}$$

$$\frac{\begin{array}{c}\text{GN} \sqsubseteq \text{G}\\ |\text{GN}|=\text{N}\\ \text{GN'} \sqsubseteq \text{G}\\ |\text{GN'}|=\text{N}\\ \text{sametenv(GN,GN')}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-SPLIT-EQ)}$$

$$\text{sub-env-size-split-eq(A1,tf/n,A2,tf/n,identtenv)}$$

$$\frac{\begin{array}{c}\text{eq2-sametenv-cons(GEQ,TEQ,EQ)}\\ \text{sub-env-size-eq-cons(GEQ,A1,A2,N1,N2,TEQ)}\\ \text{sub-env-size-split-eq(A1',N1,A2',N2,GEQ)}\\ \text{sub-env-cons(A2,A2')}\\ \text{sub-env-cons(A1,A1')}\end{array}}{\text{sub-env-size-split-eq(A1,tf/c(N1),A2,tf/c(N2),EQ)}}$$

$$\frac{\begin{array}{c}\text{GN} \sqsubseteq_{\text{N}} \text{G}\\ \text{GN'} \sqsubseteq_{\text{N}} \text{G}\\ \text{sametenv(GN,GN')}\end{array}}{\text{type}} \quad \text{(SUB-ENV-SIZE-EQ)}$$

$$\frac{\text{sub-env-size-split-eq(A1,N1,A2,N2,EQ)}}{\text{sub-env-size-eq(ses(N1,A1),ses(N2,A2),EQ)}}$$

$$\frac{\begin{array}{c}\text{sametenv(GN,GN')}\\ \text{TN::GN} \vdash \text{T} \prec_{\text{L}} \text{T2}\\ \text{TN::GN'} \vdash \text{T} \prec_{\text{L}} \text{T2}\end{array}}{\text{type}} \quad \text{(EQ-EXPAND-ENV)}$$

$$\text{eq-expand-env(identtenv,B,B)}$$

$$\frac{\begin{array}{c}\text{same(T',T)}\\ \text{G} \vdash \text{T} \prec_{\text{L}} \text{TE}\\ \text{G} \vdash \text{T'} \prec_{\text{L}} \text{TE}\end{array}}{\text{type}} \quad \text{(EQ-EXPAND)}$$

$$\text{eq-expand(ident,E,E)}$$

$$\frac{\mathrm{G} \vdash \mathrm{T} \prec_{\mathrm{LN}} \mathrm{T1} \quad \mathrm{G} \vdash \mathrm{T} \prec_{\mathrm{LN}} \mathrm{T2} \quad \mathrm{sameopt(T1,T2)}}{\mathrm{type}} \qquad \text{(EXPAND-EQ)}$$

$$\frac{\mathrm{G2} \vdash \mathrm{var(N1)} \ni_0 \{\mathbf{type}\ \ \mathrm{LN} : \mathrm{T1} \ldots \mathrm{T3}\} \quad \mathrm{G2} \vdash \mathrm{var(N2)} \ni_0 \{\mathbf{type}\ \ \mathrm{LN} : \mathrm{T2} \ldots \mathrm{T4}\} \quad \mathrm{N1} = \mathrm{N2} \quad \mathrm{same(T1,T2)} \quad \mathrm{same(T3,T4)}}{\mathrm{type}} \qquad \text{(LOOKEXP-EQ)}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/top,exp\text{-}tp/top,identopt)}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/bot,exp\text{-}tp/bot,identopt)}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/fun,exp\text{-}tp/fun,identopt)}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/recv,exp\text{-}tp/recv,identopt)}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/rect,exp\text{-}tp/rect,identopt)}$$

$$\frac{\mathrm{no\text{-}eq2(F,\{\mathbf{type}\ \ X1 : X2 \ldots X3\},\emptyset_T,S)} \quad \mathrm{ne\text{-}irrefl(N,F)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/rect,exp\text{-}tp/rectn(N),S)}}$$

$$\frac{\mathrm{no\text{-}eq2(F,\emptyset_T,\{\mathbf{type}\ \ X1 : X2 \ldots X3\},S)} \quad \mathrm{ne\text{-}irrefl(N,F)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/rectn(N),exp\text{-}tp/rect,S)}}$$

$$\mathrm{expand\text{-}eq(exp\text{-}tp/rectn(N1),exp\text{-}tp/rectn(N2),identopt)}$$

$$\frac{\mathrm{topt\text{-}and\text{-}eq(T1',T2,S)} \quad \mathrm{eq\text{-}topt\text{-}and(S1,S2,T1,T1')} \quad \mathrm{expand\text{-}eq(E2,E4,S2)} \quad \mathrm{expand\text{-}eq(E1,E3,S1)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/and(T1,E1,E2),exp\text{-}tp/and(T2,E3,E4),S)}}$$

$$\frac{\mathrm{topt\text{-}or\text{-}eq(T1',T2,S)} \quad \mathrm{eq\text{-}topt\text{-}or(S1,S2,T1,T1')} \quad \mathrm{expand\text{-}eq(E2,E4,S2)} \quad \mathrm{expand\text{-}eq(E1,E3,S1)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/or(T1,E1,E2),exp\text{-}tp/or(T2,E3,E4),S)}}$$

$$\frac{\mathrm{expand\text{-}eq(E1,E2',S)} \quad \mathrm{eq\text{-}expand\text{-}env(EQG,E2,E2')} \quad \mathrm{sub\text{-}env\text{-}size\text{-}eq(S2,S1,EQG)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/bind(E1,S1),exp\text{-}tp/bind(E2,S2),S)}}$$

$$\frac{\mathrm{expand\text{-}eq(E1,E2',S)} \quad \mathrm{eq\text{-}expand(SB,E2,E2')} \quad \mathrm{lookexp\text{-}eq(L1,L2,EQ,SA,SB)} \quad \mathrm{eq\text{-}refl(X7,EQ)}}{\mathrm{expand\text{-}eq(exp\text{-}tp/tsel(E1,L1),exp\text{-}tp/tsel(E2,L2),S)}}$$

$$\frac{\begin{array}{c}\text{eq-rect2(XX,S1,S2)}\\ \text{expand-eq(E1,E2',XX)}\\ \text{eq-expand(X,E2,E2')}\\ \text{lookup-eq0(L1,L2,E,X)}\end{array}}{\text{lookexp-eq(tle(E1,L1),tle(E2,L2),E,S1,S2)}}$$

$$\frac{\begin{array}{c}\text{sameopt(T,T')}\\ \text{same(T,T')}\end{array}}{\text{type}}\ (\textsc{sameopt-same})$$

$$\text{sameopt-same(identopt,ident)}$$

$$\frac{\begin{array}{c}\mathrm{G} \vdash \mathrm{E} \ni_0 \{\textbf{type}\ \ \mathrm{N} : \mathrm{S} \mathinner{..} \mathrm{U}\}\\ \mathrm{G} \vdash \mathrm{E} \ni_0 \{\textbf{type}\ \ \mathrm{N} : \mathrm{S'} \mathinner{..} \mathrm{U'}\}\\ \text{same}(\{\textbf{type}\ \ \mathrm{N} : \mathrm{S} \mathinner{..} \mathrm{U}\},\{\textbf{type}\ \ \mathrm{N} : \mathrm{S'} \mathinner{..} \mathrm{U'}\})\end{array}}{\text{type}}\ (\textsc{lookexp-eq-gen})$$

$$\frac{\begin{array}{c}\text{sameopt-same(S,SX)}\\ \text{expand-eq(E1',E2,S)}\\ \text{eq-exp-low(X,E1,E1')}\\ \text{lookup-eq0(L1,L2,EQ,X)}\\ \text{eq-refl(N,EQ)}\end{array}}{\text{lookexp-eq-gen(tle(E1,L1),tle(E2,L2),SX)}}$$

$$\frac{\begin{array}{c}\text{false}\\ \text{M is mode}\\ \text{G1 is tenv}\\ \text{T1 is tpe}\\ \text{G2 is tenv}\\ \text{T2 is tpe}\\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\end{array}}{\text{type}}\ (\textsc{no-subtype})$$

$$\frac{\begin{array}{c}\text{false}\\ \text{M is mode}\\ \text{G1 is tenv}\\ \text{T1 is tpe}\\ \mathrm{G1} \vdash \mathrm{T1}\ \mathbf{wf_M}\end{array}}{\text{type}}\ (\textsc{no-wf-tp})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{T1}\ \mathbf{wf_M}\\ \text{Z is tpe}\\ \mathrm{Z::G1} \vdash \mathrm{T1}\ \mathbf{wf_M}\end{array}}{\text{type}}\ (\textsc{extend-wf-tp})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{E} \ni_{\mathrm{M}} \mathrm{T1}\\ \text{Z is tpe}\\ \mathrm{Z::G1} \vdash \mathrm{E} \ni_{\mathrm{M}} \mathrm{T1}\end{array}}{\text{type}}\ (\textsc{extend-wf-pev})$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\\ \text{Z is tpe}\\ \mathrm{Z::G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{Z::G2}\\ \mathrm{Z::G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2}\\ \mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{Z::G2}\end{array}}{\text{type}}\ (\textsc{extend-sub-tp})$$

$$\frac{\text{extend-wf-lkpe(L,T,L1)}}{\text{extend-wf-pev(pev(L),T,pev(L1))}}$$

$$\text{extend-wf-tp(wf-tp/top,T,wf-tp/top)}$$

$$\text{extend-wf-tp(wf-tp/bot,T,wf-tp/bot)}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W2,T,W2')}\\ \text{extend-wf-tp(W1,T,W1')}\end{array}}{\text{extend-wf-tp(wf-tp/fun(W1,W2),T,wf-tp/fun(W1',W2'))}}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W2,T,W2')}\\ \text{extend-wf-tp(W1,T,W1')}\\ \text{extend-sub-tp(ST,T,ST',X5,X6)}\end{array}}{\text{extend-wf-tp(wf-tp/rect(ST,W1,W2),T,wf-tp/rect(ST',W1',W2'))}}$$

$$\frac{\text{extend-wf-tp(W1,T,W1')}}{\text{extend-wf-tp(wf-tp/recv(W1),T,wf-tp/recv(W1'))}}$$

$$\frac{\text{extend-wf-pev(P,T,P')}}{\text{extend-wf-tp(wf-tp/tsel(W,ses(N,S),P),T,wf-tp/tsel(W,ses(N,sub-env/ext(S)),P'))}}$$

$$\text{extend-wf-tp(wf-tp/tbind(W,ses(N,S)),T,wf-tp/tbind(W,ses(N,sub-env/ext(S))))}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W2,T,W2')}\\ \text{extend-wf-tp(W1,T,W1')}\end{array}}{\text{extend-wf-tp(wf-tp/and(W1,W2),T,wf-tp/and(W1',W2'))}}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W2,T,W2')}\\ \text{extend-wf-tp(W1,T,W1')}\end{array}}{\text{extend-wf-tp(wf-tp/or(W1,W2),T,wf-tp/or(W1',W2'))}}$$

$$\frac{\text{extend-wf-tp(ST,T,ST')}}{\text{extend-sub-tp(sub-tp/top(ST),T,sub-tp/top(ST'),sub-tp/top(ST'),sub-tp/top(ST))}}$$

$$\frac{\text{extend-wf-tp(ST,T,ST')}}{\text{extend-sub-tp(sub-tp/bot(ST),T,sub-tp/bot(ST'),sub-tp/bot(ST),sub-tp/bot(ST'))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST2,T,ST2',ST2A,ST2B)}\\ \text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}\end{array}}{\text{extend-sub-tp(sub-tp/fun(ST1,ST2),T,sub-tp/fun(ST1',ST2'),sub-tp/fun(ST1A,ST2B),sub-tp/fun(ST1B,ST2A))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST2,T,ST2',ST2A,ST2B)}\\ \text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}\\ \text{extend-sub-tp(B1,T,B1',X8,X9)}\\ \text{extend-sub-tp(B2,T,B2',X10,X11)}\end{array}}{\text{extend-sub-tp(sub-tp/rect(ST1,ST2,B2,B1),T,sub-tp/rect(ST1',ST2',B2',B1'),sub-tp/rect(ST1A,ST2B,B2,B1'),sub-tp/rect(ST}$$

$$\frac{\text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}}{\text{extend-sub-tp(sub-tp/recv(ST1),T,sub-tp/recv(ST1'),sub-tp/recv(ST1A),sub-tp/recv(ST1B))}}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W,T,W')}\\ \text{extend-sub-tp(ST,T,X9,X10,ST')}\\ \text{extend-wf-pev(P,T,P')}\end{array}}{\text{extend-sub-tp(sub-tp/tsel1(W,ST,ses(N,S),P),T,sub-tp/tsel1(W',ST',ses(N,sub-env/ext(S)),P'),sub-tp/tsel1(W',ST,ses(N,sub-e}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W,T,W')}\\ \text{extend-sub-tp(ST,T,X8,ST',X9)}\\ \text{extend-wf-pev(P,T,P')}\end{array}}{\text{extend-sub-tp(sub-tp/tsel2-strict(W,ST,ses(N,S),P),T,sub-tp/tsel2-strict(W',ST',ses(N,sub-env/ext(S)),P'),sub-tp/tsel2-strict(}}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W,T,W')}\\ \text{extend-sub-tp(ST,T,X8,ST',X9)}\\ \text{extend-wf-pev(P,T,P')}\end{array}}{\text{extend-sub-tp(sub-tp/tsel2-lenient(W,ST,ses(N,S),P),T,sub-tp/tsel2-lenient(W',ST',ses(N,sub-env/ext(S)),P'),sub-tp/tsel2-leni}}$$

$$\frac{}{\text{extend-sub-tp(sub-tp/tselx-strict(ST,ses(N2,S2),ses(N1,S1),P2,P1),T,sub-tp/tselx-strict(ST,ses(N2,sub-env/ext(S2)),ses(N1,sub}}$$

$$\frac{\text{extend-sub-tp(ST,T,STA1,STB1,STC1)}}{\text{extend-sub-tp(sub-tp/tbind1(ST,ses(N,S)),T,sub-tp/tbind1(STC1,ses(N,sub-env/ext(S))),sub-tp/tbind1(ST,ses(N,sub-env/ext(}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST3,T,STA3,STB3,STC3)}\\ \text{extend-sub-tp(ST2,T,STA2,STB2,STC2)}\\ \text{extend-sub-tp(ST1,T,STA1,STB1,STC1)}\end{array}}{\text{extend-sub-tp(sub-tp/tbind2(ST1,ST2,ST3,ses(N,S)),T,sub-tp/tbind2(STB1,STB2,STB3,ses(N,sub-env/ext(S))),sub-tp/tbind2}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}\\ \text{extend-wf-tp(W2,T,W2')}\end{array}}{\text{extend-sub-tp(sub-tp/and1a(W2,ST1),T,sub-tp/and1a(W2',ST1'),sub-tp/and1a(W2',ST1A),sub-tp/and1a(W2,ST1B))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST2,T,ST2',ST2A,ST2B)}\\ \text{extend-wf-tp(W1,T,W1')}\end{array}}{\text{extend-sub-tp(sub-tp/and1b(W1,ST2),T,sub-tp/and1b(W1',ST2'),sub-tp/and1b(W1',ST2A),sub-tp/and1b(W1,ST2B))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST2,T,ST2',ST2A,ST2B)}\\ \text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}\end{array}}{\text{extend-sub-tp(sub-tp/and2(ST1,ST2),T,sub-tp/and2(ST1',ST2'),sub-tp/and2(ST1A,ST2A),sub-tp/and2(ST1B,ST2B))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST,T,ST',STA,STB)}\\ \text{extend-wf-tp(W2,T,W2')}\end{array}}{\text{extend-sub-tp(sub-tp/or1a(W2,ST),T,sub-tp/or1a(W2',ST'),sub-tp/or1a(W2,STA),sub-tp/or1a(W2',STB))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST,T,ST',STA,STB)}\\ \text{extend-wf-tp(W2,T,W2')}\end{array}}{\text{extend-sub-tp(sub-tp/or1b(W2,ST),T,sub-tp/or1b(W2',ST'),sub-tp/or1b(W2,STA),sub-tp/or1b(W2',STB))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST2,T,ST2',ST2A,ST2B)}\\ \text{extend-sub-tp(ST1,T,ST1',ST1A,ST1B)}\end{array}}{\text{extend-sub-tp(sub-tp/or2(ST1,ST2),T,sub-tp/or2(ST1',ST2'),sub-tp/or2(ST1A,ST2A),sub-tp/or2(ST1B,ST2B))}}$$

$$\text{extend-sub-tp(sub-tp/refl(W,S2,S1),T,sub-tp/refl(W,sub-env/ext(S2),sub-env/ext(S1)),sub-tp/refl(W,S2,sub-env/ext(S1)),sub-}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{Z is tpe}\\ \text{Z::G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{Z::G2}\end{array}}{\text{type}} \qquad (\textsc{extend-sub-tp}12)$$

$$\frac{\text{extend-sub-tp(ST,Z,ST',X6,X7)}}{\text{extend-sub-tp12(ST,Z,ST')}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2} \\ \text{Z is tpe} \\ \text{Z::G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad (\textsc{extend-sub-tp1})$$

$$\frac{\text{extend-sub-tp(ST,Z,X6,ST',X7)}}{\text{extend-sub-tp1(ST,Z,ST')}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2} \\ \text{Z is tpe} \\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{Z::G2}\end{array}}{\text{type}} \quad (\textsc{extend-sub-tp2})$$

$$\frac{\text{extend-sub-tp(ST,Z,X6,X7,ST')}}{\text{extend-sub-tp2(ST,Z,ST')}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} \ \mathbf{wf_M} \\ \text{G1} \sqsubseteq \text{G2} \\ \text{G2} \vdash \text{T1} \ \mathbf{wf_M}\end{array}}{\text{type}} \quad (\textsc{extend-wf-tp-mult})$$

$$\text{extend-wf-tp-mult(W,sub-env/refl,W)}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp(W1,X4,W2)} \\ \text{extend-wf-tp-mult(W,S,W1)}\end{array}}{\text{extend-wf-tp-mult(W,sub-env/ext(S),W2)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2} \\ \text{G1} \sqsubseteq \text{G3} \\ \text{G3} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad (\textsc{extend-sub-tp1-mult})$$

$$\text{extend-sub-tp1-mult(ST,sub-env/refl,ST)}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp1(ST1,X6,ST2)} \\ \text{extend-sub-tp1-mult(ST,S,ST1)}\end{array}}{\text{extend-sub-tp1-mult(ST,sub-env/ext(S),ST2)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2} \\ \text{G2} \sqsubseteq \text{G3} \\ \text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G3}\end{array}}{\text{type}} \quad (\textsc{extend-sub-tp2-mult})$$

$$\text{extend-sub-tp2-mult(ST,sub-env/refl,ST)}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp2(ST1,X6,ST2)} \\ \text{extend-sub-tp2-mult(ST,S,ST1)}\end{array}}{\text{extend-sub-tp2-mult(ST,sub-env/ext(S),ST2)}}$$

$$\frac{\begin{array}{c}\mathrm{G} \vdash \mathrm{E} \ni_{\mathrm{M}} \mathrm{T} \\ \mathrm{G} \sqsubseteq \mathrm{G1} \\ \mathrm{G1} \vdash \mathrm{E} \ni_{\mathrm{M}} \mathrm{T}\end{array}}{\text{type}} \quad \text{(EXTEND-WF-PEV-MULT)}$$

$$\text{extend-wf-pev-mult(P,sub-env/refl,P)}$$

$$\frac{\begin{array}{c}\text{extend-wf-pev(P1,X5,P2)} \\ \text{extend-wf-pev-mult(P,S,P1)}\end{array}}{\text{extend-wf-pev-mult(P,sub-env/ext(S),P2)}}$$

$$\frac{\begin{array}{c}\text{tenv} \\ \text{tenv} \\ \text{tpe} \\ \text{tenv} \\ \text{tenv}\end{array}}{\text{type}} \quad \text{(SUB-TP-ENV)}$$

$$\text{sub-tp-env(X1,X2,X3,G,G)} \quad \text{(STE/N)}$$

$$\frac{\begin{array}{c}\text{sub-tp-env(GX,GY,T3,G1,G2)} \\ \mathrm{GX} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{GY} \\ \mathrm{GY} \vdash \mathrm{T2} <:_{\mathrm{M}} \mathrm{T3} \dashv \mathrm{T2::G2}\end{array}}{\text{sub-tp-env(GX,GY,T3,T1::G1,T2::G2)}} \quad \text{(STE/C)}$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \mathrm{T1} <:_{\mathrm{M}} \mathrm{T2} \dashv \mathrm{G2} \\ \mathrm{G1} \vdash \mathrm{T1}\ \mathbf{wf_M} \\ \mathrm{G2} \vdash \mathrm{T2}\ \mathbf{wf_M}\end{array}}{\text{type}} \quad \text{(EXTRACT-WF)}$$

$$\text{extract-wf(sub-tp/top(W),W,wf-tp/top)}$$

$$\text{extract-wf(sub-tp/bot(W),wf-tp/bot,W)}$$

$$\frac{\begin{array}{c}\text{extract-wf(ST2,W3,W1)} \\ \text{extract-wf(ST1,W2,W4)}\end{array}}{\text{extract-wf(sub-tp/fun(ST1,ST2),wf-tp/fun(W2,W1),wf-tp/fun(W4,W3))}}$$

$$\frac{\begin{array}{c}\text{extract-wf(ST2,W3,W1)} \\ \text{extract-wf(ST1,W2,W4)}\end{array}}{\text{extract-wf(sub-tp/rect(ST1,ST2,B2,B1),wf-tp/rect(B1,W2,W1),wf-tp/rect(B2,W4,W3))}}$$

$$\frac{\text{extract-wf(ST,W1,W2)}}{\text{extract-wf(sub-tp/recv(ST),wf-tp/recv(W1),wf-tp/recv(W2))}}$$

$$\frac{\text{extract-wf(ST,X6,W2)}}{\text{extract-wf(sub-tp/tsel1(W1,ST,G,P),W1,W2)}}$$

$$\frac{\text{extract-wf(ST,W1,X5)}}{\text{extract-wf(sub-tp/tsel2-strict(W2,ST,G,P),W1,W2)}}$$

$$\frac{\text{extract-wf(ST,W1,X5)}}{\text{extract-wf(sub-tp/tsel2-lenient(W2,ST,G,P),W1,W2)}}$$

$$\frac{\text{extract-wf(ST,W1,W2)}}{\text{extract-wf(sub-tp/tselx-strict(ST,G2,G1,P2,P1),wf-tp/tsel(W1,G1,P1),wf-tp/tsel(W2,G2,P2))}}$$

$$\frac{\text{extract-wf(ST,WU,W2)}}{\text{extract-wf(sub-tp/tbind1(ST,S),wf-tp/tbind(WU,S),W2)}}$$

$$\frac{\text{extract-wf(SU,W1,W2)}}{\text{extract-wf(sub-tp/tbind2(SW,SE,SU,S),W1,wf-tp/tbind(W2,S))}}$$

$$\frac{\text{extract-wf(S1,W1,W3)}}{\text{extract-wf(sub-tp/and1a(W2,S1),wf-tp/and(W1,W2),W3)}}$$

$$\frac{\text{extract-wf(S2,W2,W3)}}{\text{extract-wf(sub-tp/and1b(W1,S2),wf-tp/and(W1,W2),W3)}}$$

$$\frac{\begin{array}{c}\text{extract-wf(S2,W2,W4)}\\ \text{extract-wf(S1,W1,W3)}\end{array}}{\text{extract-wf(sub-tp/and2(S1,S2),W1,wf-tp/and(W3,W4))}}$$

$$\frac{\text{extract-wf(S1,W3,W1)}}{\text{extract-wf(sub-tp/or1a(W2,S1),W3,wf-tp/or(W1,W2))}}$$

$$\frac{\text{extract-wf(S2,W3,W2)}}{\text{extract-wf(sub-tp/or1b(W1,S2),W3,wf-tp/or(W1,W2))}}$$

$$\frac{\begin{array}{c}\text{extract-wf(S2,W2,W4)}\\ \text{extract-wf(S1,W1,W3)}\end{array}}{\text{extract-wf(sub-tp/or2(S1,S2),wf-tp/or(W1,W2),W3)}}$$

$$\frac{\begin{array}{c}\text{extend-wf-tp-mult(W,S2,W2)}\\ \text{extend-wf-tp-mult(W,S1,W1)}\end{array}}{\text{extract-wf(sub-tp/refl(W,S2,S1),W1,W2)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T } \mathbf{wf_M}\\ \text{G2} \vdash \text{T } \mathbf{wf_M}\\ \text{minmax-env(G1,G2,GS,GU)}\\ \text{GS} \vdash \text{T } \mathbf{wf_M}\\ \text{GS} \sqsubseteq \text{G1}\\ \text{GS} \sqsubseteq \text{G2}\end{array}}{\text{type}} \quad \text{(WF-TP-MIN)}$$

wf-tp-min(W1,W2,mm-env/1(S12),W1,sub-env/refl,S12)

wf-tp-min(W1,W2,mm-env/2(S21),W2,S21,sub-env/refl)

$$\frac{\begin{array}{c}\text{GT} \vdash \text{T } \mathbf{wf_M}\\ \text{G1} \vdash \text{T } \mathbf{wf_M}\\ \text{G2} \vdash \text{T } \mathbf{wf_M}\\ \text{G1} \sqsubseteq \text{GT}\\ \text{G2} \sqsubseteq \text{GT}\\ \text{G} \vdash \text{T } \mathbf{wf_M}\\ \text{G} \sqsubseteq \text{G1}\\ \text{G} \sqsubseteq \text{G2}\end{array}}{\text{type}} \quad \text{(WF-TP-DIA)}$$

$$\frac{\begin{array}{c}\text{wf-tp-min(W1,W2,MM,W0,S01,S02)}\\ \text{sub-env-dia(S1,S2,MM)}\end{array}}{\text{wf-tp-dia(WT,W1,W2,S1,S2,W0,S01,S02)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G2} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{minmax-env(G1,G2,GS,GU)}\\ \text{GS} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{GS} \sqsubseteq \text{G1}\\ \text{GS} \sqsubseteq \text{G2}\end{array}}{\text{type}} \quad (\textsc{pev-tp-min})$$

$$\text{pev-tp-min(W1,W2,mm-env/1(S12),W1,sub-env/refl,S12)}$$

$$\text{pev-tp-min(W1,W2,mm-env/2(S21),W2,S21,sub-env/refl)}$$

$$\frac{\begin{array}{c}\text{GT} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G1} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G2} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G1} \sqsubseteq \text{GT}\\ \text{G2} \sqsubseteq \text{GT}\\ \text{G} \vdash \text{N} \ni_{\text{M}} \text{T}\\ \text{G} \sqsubseteq \text{G1}\\ \text{G} \sqsubseteq \text{G2}\end{array}}{\text{type}} \quad (\textsc{pev-tp-dia})$$

$$\frac{\begin{array}{c}\text{pev-tp-min(W1,W2,MM,W0,S01,S02)}\\ \text{sub-env-dia(S1,S2,MM)}\end{array}}{\text{pev-tp-dia(WT,W1,W2,S1,S2,W0,S01,S02)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{T}\ \mathbf{wf}_{\approx}\\ \text{G} \vdash \text{T} <:_{\approx} \text{T} \dashv \text{G}\end{array}}{\text{type}} \quad (\textsc{sub-tp-refl})$$

$$\text{sub-tp-refl(wf-tp/bot,sub-tp/bot(wf-tp/bot))}$$

$$\text{sub-tp-refl(wf-tp/top,sub-tp/top(wf-tp/top))}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(T2,SBT2)}\\ \text{sub-tp-refl(T1,SBT1)}\end{array}}{\text{sub-tp-refl(wf-tp/fun(T1,T2),sub-tp/fun(SBT1,SBT2))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(T2,SBT2)}\\ \text{sub-tp-refl(T1,SBT1)}\end{array}}{\text{sub-tp-refl(wf-tp/rect(ST,T1,T2),sub-tp/rect(SBT1,SBT2,ST,ST))}}$$

$$\frac{\text{sub-tp-refl(T1,SBT1)}}{\text{sub-tp-refl(wf-tp/recv(T1),sub-tp/recv(SBT1))}}$$

$$\frac{\text{sub-tp-refl(W,sub-tp/rect(BU,X5,X6,X7))}}{\text{sub-tp-refl(wf-tp/tsel(W,G,P),sub-tp/tsel1(wf-tp/tsel(W,G,P),sub-tp/tsel2-lenient(wf-tp/tsel(W,G,P),BU,G,P),G,P))}}$$

$$\frac{\text{sub-tp-refl(WU,R)}}{\text{sub-tp-refl(wf-tp/tbind(WU,S),sub-tp/tbind1(sub-tp/tbind2(R,R,R,S),S))}}$$

$$\frac{\text{sub-tp-refl(W2,ST2)} \quad \text{sub-tp-refl(W1,ST1)}}{\text{sub-tp-refl(wf-tp/and(W1,W2),sub-tp/and2(sub-tp/and1a(W2,ST1),sub-tp/and1b(W1,ST2)))}}$$

$$\frac{\text{sub-tp-refl(W2,ST2)} \quad \text{sub-tp-refl(W1,ST1)}}{\text{sub-tp-refl(wf-tp/or(W1,W2),sub-tp/or2(sub-tp/or1a(W2,ST1),sub-tp/or1b(W1,ST2)))}}$$

$$\frac{\mathrm{G} \vdash \mathrm{E} \ni_{\mathrm{M}} \{\textbf{type}\ \ \mathrm{N} : \mathrm{S} \ldots \mathrm{U}\} \quad \mathrm{G} \vdash \mathrm{E} \ni_{\mathrm{M}} \{\textbf{type}\ \ \mathrm{N} : \mathrm{S'} \ldots \mathrm{U'}\} \quad \text{same}(\{\textbf{type}\ \ \mathrm{N} : \mathrm{S} \ldots \mathrm{U}\},\{\textbf{type}\ \ \mathrm{N} : \mathrm{S'} \ldots \mathrm{U'}\})}{\text{type}} \quad \text{(PATH-EVAL-EQ)}$$

$$\frac{\text{lookexp-eq-gen(L,L',EQT)}}{\text{path-eval-eq(pev(L),pev(L'),EQT)}}$$

$$\frac{\text{sametenv(G1,G2)} \quad \mathrm{G1} \sqsubseteq \mathrm{G2}}{\text{type}} \quad \text{(SAMETENV-SUB-ENV)}$$

$$\text{sametenv-sub-env(identtenv,sub-env/refl)}$$

$$\frac{\text{sametenv(G,GX)} \quad \mathrm{G} \sqsubseteq \mathrm{G2} \quad \mathrm{GX} \sqsubseteq \mathrm{G2}}{\text{type}} \quad \text{(EQ-SUB-ENV-LOW)}$$

$$\text{eq-sub-env-low(identtenv,S,S)}$$

$$\frac{\text{sametenv(G,GX)} \quad \mathrm{G1} \sqsubseteq \mathrm{G} \quad \mathrm{G1} \sqsubseteq \mathrm{GX}}{\text{type}} \quad \text{(EQ-SUB-ENV-HIGH)}$$

$$\text{eq-sub-env-high(identtenv,S,S)}$$

$$\frac{\text{same}(\{\textbf{type}\ \ \mathrm{N} : \mathrm{X1} \ldots \mathrm{X2}\},\{\textbf{type}\ \ \mathrm{N} : \mathrm{X1'} \ldots \mathrm{X2'}\}) \quad \text{same(X1,X1')} \quad \text{same(X2,X2')}}{\text{type}} \quad \text{(RECT-SAME-ALL-SAME)}$$

$$\text{rect-same-all-same(ident,ident,ident)}$$

$$\frac{\text{sametenv(G,G')} \quad |\mathrm{G}|{=}\mathrm{S} \quad |\mathrm{G'}|{=}\mathrm{S}}{\text{type}} \quad \text{(EQ-TSIZE-ENV)}$$

$$\text{eq-tsize-env(identtenv,S,S)}$$

$$\frac{\mathrm{G2} \vdash \mathrm{T2}\ \mathbf{wf}_{\approx} \quad \mathrm{G1} \vdash \mathrm{T1} <:_{\approx} \mathrm{T2} \dashv \mathrm{G2} \quad \mathrm{G2} \vdash \mathrm{T2} <:_{\approx} \mathrm{T3} \dashv \mathrm{G3} \quad \mathrm{G1} \vdash \mathrm{T1} <:_{\approx} \mathrm{T3} \dashv \mathrm{G3}}{\text{type}} \quad \text{(SUB-TP-TRANS-AUX)}$$

$$\frac{\text{extract-wf(S1,W,X5)}}{\text{sub-tp-trans-aux(X7,S1,sub-tp/top(X8),sub-tp/top(W))}}$$

$$\frac{\text{extract-wf(S2,X5,W)}}{\text{sub-tp-trans-aux(X7,sub-tp/bot(X8),S2,sub-tp/bot(W))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(T2,SBT2,SBT4,SBT6)}\\ \text{sub-tp-trans-aux(T1,SBT3,SBT1,SBT5)}\end{array}}{\text{sub-tp-trans-aux(wf-tp/fun(T2,T1),sub-tp/fun(SBT2,SBT1),sub-tp/fun(SBT4,SBT3),sub-tp/fun(SBT6,SBT5))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(T2,SBT2,SBT4,SBT6)}\\ \text{sub-tp-trans-aux(T1,SBT3,SBT1,SBT5)}\end{array}}{\text{sub-tp-trans-aux(wf-tp/rect(SB,T2,T1),sub-tp/rect(SBT2,SBT1,B2,B1),sub-tp/rect(SBT4,SBT3,B4,B3),sub-tp/rect(SBT6,SB}$$

$$\frac{\text{sub-tp-trans-aux(T1,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(wf-tp/recv(T1),sub-tp/recv(SBT1),sub-tp/recv(SBT3),sub-tp/recv(SBT5))}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(W,sub-tp/tsel1(W1,SBT1,G,P),SBT3,sub-tp/tsel1(W1,SBT5,G,P))}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(W,SBT1,sub-tp/tsel2-lenient(W2,SBT3,G,P),sub-tp/tsel2-lenient(W2,SBT5,G,P))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(WU,A1”,A2”,OUT)}\\ \text{eq-low-env(EQG2W,A2',A2”)}\\ \text{eq-low(EQU2W,A2,A2')}\\ \text{eq-high-env(EQG1W,A1',A1”)}\\ \text{eq-high(EQU1W,A1,A1')}\\ \text{sub-env-size-eq(GE2,GEW,EQG2W)}\\ \text{rect-same-all-same(EQR2W,EQS2W,EQU2W)}\\ \text{path-eval-eq(P2,PW,EQR2W)}\\ \text{sub-env-size-eq(GE1,GEW,EQG1W)}\\ \text{rect-same-all-same(EQR1W,EQS1W,EQU1W)}\\ \text{path-eval-eq(P1,PW,EQR1W)}\\ \text{sub-env-size-eq(GE1,GE2,EQG12)}\\ \text{rect-same-all-same(EQR12,EQS12,EQU12)}\\ \text{path-eval-eq(P1,P2,EQR12)}\end{array}}{\text{sub-tp-trans-aux(wf-tp/tsel(wf-tp/rect(BSU,WU,WS),GEW,PW),sub-tp/tsel2-lenient(X17,A1,GE1,P1),sub-tp/tsel1(X18,A2,G}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(W,SBT1,SBT3,SBT5)}\\ \text{sub-tp-trans-aux(W,SBT1,SU,SU')}\\ \text{sub-tp-trans-aux(W,SBT1,SW,SW')}\end{array}}{\text{sub-tp-trans-aux(W,SBT1,sub-tp/tbind2(SW,SBT3,SU,S),sub-tp/tbind2(SW',SBT5,SU',S))}}$$

$$\frac{\text{sub-tp-trans-aux(W,ST,SBT2,ST')}}{\text{sub-tp-trans-aux(W,sub-tp/tbind1(ST,S),SBT2,sub-tp/tbind1(ST',S))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(WU0,SU',A2',A3)}\\ \text{eq-low-env(EQG2',A2,A2')}\\ \text{eq-sametenv-cons(EQG2,\{ X1 } \Rightarrow \text{ X2 \},EQG2')}\\ \text{sub-env-size-eq(S2,S0,EQG2)}\\ \text{eq-high-env(EQG1',SU,SU')}\\ \text{eq-sametenv-cons(EQG1,\{ X1 } \Rightarrow \text{ X2 \},EQG1')}\\ \text{sub-env-size-eq(S1,S0,EQG1)}\end{array}}{\text{sub-tp-trans-aux(wf-tp/tbind(WU0,S0),sub-tp/tbind2(X13,X14,SU,S1),sub-tp/tbind1(A2,S2),A3)}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(W,sub-tp/and1a(W1,SBT1),SBT3,sub-tp/and1a(W1,SBT5))}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(W,sub-tp/and1b(W1,SBT1),SBT3,sub-tp/and1b(W1,SBT5))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(W,SBT1,ST2,ST2')}\\ \text{sub-tp-trans-aux(W,SBT1,ST1,ST1')}\end{array}}{\text{sub-tp-trans-aux(W,SBT1,sub-tp/and2(ST1,ST2),sub-tp/and2(ST1',ST2'))}}$$

$$\frac{\text{sub-tp-trans-aux(W1,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(wf-tp/and(W1,W2),sub-tp/and2(SBT1,X8),sub-tp/and1a(X9,SBT3),SBT5)}}$$

$$\frac{\text{sub-tp-trans-aux(W2,SBT1,SBT3,SBT5)}}{\text{sub-tp-trans-aux(wf-tp/and(W1,W2),sub-tp/and2(X8,SBT1),sub-tp/and1b(X9,SBT3),SBT5)}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT3,SBT1,SBT5)}}{\text{sub-tp-trans-aux(W,SBT3,sub-tp/or1a(W1,SBT1),sub-tp/or1a(W1,SBT5))}}$$

$$\frac{\text{sub-tp-trans-aux(W,SBT3,SBT1,SBT5)}}{\text{sub-tp-trans-aux(W,SBT3,sub-tp/or1b(W1,SBT1),sub-tp/or1b(W1,SBT5))}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(W,ST2,SBT1,ST2')}\\ \text{sub-tp-trans-aux(W,ST1,SBT1,ST1')}\end{array}}{\text{sub-tp-trans-aux(W,sub-tp/or2(ST1,ST2),SBT1,sub-tp/or2(ST1',ST2'))}}$$

$$\frac{\text{sub-tp-trans-aux(W1,SBT3,SBT1,SBT5)}}{\text{sub-tp-trans-aux(wf-tp/or(W1,W2),sub-tp/or1a(X8,SBT3),sub-tp/or2(SBT1,X9),SBT5)}}$$

$$\frac{\text{sub-tp-trans-aux(W2,SBT3,SBT1,SBT5)}}{\text{sub-tp-trans-aux(wf-tp/or(W1,W2),sub-tp/or1b(X8,SBT3),sub-tp/or2(X9,SBT1),SBT5)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\approx} \text{T2} \dashv \text{G2}\\ \text{G2} \vdash \text{T2} <:_{\approx} \text{T3} \dashv \text{G3}\\ \text{G1} \vdash \text{T1} <:_{\approx} \text{T3} \dashv \text{G3}\end{array}}{\text{type}} \quad \text{(SUB-TP-TRANS)}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans-aux(T2,SBT1,SBT2,SBT3)}\\ \text{extract-wf(SBT1,T1,T2)}\end{array}}{\text{sub-tp-trans(SBT1,SBT2,SBT3)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{T } \mathbf{wf}_{!}\\ \text{G} \vdash \text{T } \mathbf{wf}_{\approx}\end{array}}{\text{type}} \quad \text{(WIDEN-MODE-WF-TP)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{E} \ni_{!} \text{T}\\ \text{G} \vdash \text{E} \ni_{\approx} \text{T}\end{array}}{\text{type}} \quad \text{(WIDEN-MODE-PATH-EVAL)}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{!} \text{T2} \dashv \text{G2}\\ \text{G1} \vdash \text{T1} <:_{\approx} \text{T2} \dashv \text{G2}\end{array}}{\text{type}} \quad \text{(WIDEN-MODE-SUB-TP)}$$

widen-mode-wf-tp(wf-tp/top,wf-tp/top)

widen-mode-wf-tp(wf-tp/bot,wf-tp/bot)

$$\frac{\text{widen-mode-wf-tp(B,B')} \quad \text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/fun(B,A),wf-tp/fun(B',A'))}}$$

$$\frac{\text{widen-mode-sub-tp(ST,ST')} \quad \text{widen-mode-wf-tp(B,B')} \quad \text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/rect(ST,B,A),wf-tp/rect(ST',B',A'))}}$$

$$\frac{\text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/recv(A),wf-tp/recv(A'))}}$$

$$\frac{\text{widen-mode-wf-tp(W,W')} \quad \text{widen-mode-path-eval(P,P')}}{\text{widen-mode-wf-tp(wf-tp/tsel(W,G,P),wf-tp/tsel(W',G,P'))}}$$

$$\frac{\text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/tbind(A,N),wf-tp/tbind(A',N))}}$$

$$\frac{\text{widen-mode-wf-tp(B,B')} \quad \text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/and(B,A),wf-tp/and(B',A'))}}$$

$$\frac{\text{widen-mode-wf-tp(B,B')} \quad \text{widen-mode-wf-tp(A,A')}}{\text{widen-mode-wf-tp(wf-tp/or(B,A),wf-tp/or(B',A'))}}$$

widen-mode-path-eval(pev(L),pev(L))

$$\frac{\text{widen-mode-sub-tp(B,B')} \quad \text{widen-mode-sub-tp(A,A')}}{\text{widen-mode-sub-tp(sub-tp/fun(B,A),sub-tp/fun(B',A'))}}$$

$$\frac{\text{widen-mode-sub-tp(D,D')} \quad \text{widen-mode-sub-tp(C,C')} \quad \text{widen-mode-sub-tp(B,B')} \quad \text{widen-mode-sub-tp(A,A')}}{\text{widen-mode-sub-tp(sub-tp/rect(D,C,B,A),sub-tp/rect(D',C',B',A'))}}$$

$$\frac{\text{widen-mode-sub-tp(A,A')}}{\text{widen-mode-sub-tp(sub-tp/recv(A),sub-tp/recv(A'))}}$$

$$\frac{\begin{array}{c}\text{extract-wf(sub-tp/rect(BU,BS,B2,B1),W1',W2')} \\ \text{widen-mode-sub-tp(A,sub-tp/rect(BU,BS,B2,B1))} \\ \text{widen-mode-path-eval(P2,P2')} \\ \text{widen-mode-path-eval(P1,P1')}\end{array}}{\text{widen-mode-sub-tp(sub-tp/tselx-strict(A,G2,G1,P2,P1),sub-tp/tsel1(wf-tp/tsel(W1',G1,P1'),sub-tp/tsel2-lenient(wf-tp/tsel(W}$$

widen-mode-sub-tp(C,C')
widen-mode-sub-tp(B,B')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/tbind2(C,B,A,N),sub-tp/tbind2(C',B',A',N))

widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/tbind1(A,N),sub-tp/tbind1(A',N))

widen-mode-sub-tp(B,B')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/and2(B,A),sub-tp/and2(B',A'))

widen-mode-wf-tp(W,W')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/and1a(W,A),sub-tp/and1a(W',A'))

widen-mode-wf-tp(W,W')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/and1b(W,A),sub-tp/and1b(W',A'))

widen-mode-sub-tp(B,B')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/or2(B,A),sub-tp/or2(B',A'))

widen-mode-wf-tp(W,W')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/or1a(W,A),sub-tp/or1a(W',A'))

widen-mode-wf-tp(W,W')
widen-mode-sub-tp(A,A')
---
widen-mode-sub-tp(sub-tp/or1b(W,A),sub-tp/or1b(W',A'))

widen-mode-wf-tp(W,W')
widen-mode-sub-tp(A,A')
widen-mode-path-eval(P,P')
---
widen-mode-sub-tp(sub-tp/tsel1(W,A,G,P),sub-tp/tsel1(W',A',G,P'))

widen-mode-wf-tp(wf-tp/tsel(wf-tp/rect(BSU,WU,WS),G0,P0),W')
sub-tp-trans(AS',BSU'4,A')
eq-high-env(EQG,BSU'3,BSU'4)
eq-high(EQU,BSU'2,BSU'3)
eq-low-env(EQG,BSU'1,BSU'2)
eq-low(EQS,BSU',BSU'1)
sub-env-size-eq(G0,G,EQG)
rect-same-all-same(EQR,EQS,EQU)
path-eval-eq(P0,P,EQR)
widen-mode-sub-tp(BSU,BSU')
widen-mode-sub-tp(A,AS')
widen-mode-path-eval(P,P')
---
widen-mode-sub-tp(sub-tp/tsel2-strict(wf-tp/tsel(wf-tp/rect(BSU,WU,WS),G0,P0),A,G,P),sub-tp/tsel2-lenient(W',A',G,P'))

widen-mode-wf-tp(W,W')
---
widen-mode-sub-tp(sub-tp/top(W),sub-tp/top(W'))

$$\frac{\text{widen-mode-wf-tp(W,W')}}{\text{widen-mode-sub-tp(sub-tp/bot(W),sub-tp/bot(W'))}}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp2-mult(ST',S2,ST'')}\\ \text{extend-sub-tp1-mult(ST,S1,ST')}\\ \text{sub-tp-refl(W',ST)}\\ \text{widen-mode-wf-tp(W,W')}\end{array}}{\text{widen-mode-sub-tp(sub-tp/refl(W,S2,S1),ST'')}}$$

$$\frac{\begin{array}{c}\text{tpe}\\ \text{tpe}\end{array}}{\text{type}} \quad \text{(INCOMPAT-TP)}$$

$$\text{incompat-tp}(\top,\{\textbf{val}\ \ \text{X1} : \text{X2}\}) \quad \text{(INCOMPAT/TOP-RECV)}$$

$$\text{incompat-tp}(\top,\{\textbf{def}\ \ \text{X1} : \text{X2} \to \text{X3}\}) \quad \text{(INCOMPAT/TOP-ARROW)}$$

$$\text{incompat-tp}(\{\textbf{val}\ \ \text{X1} : \text{X2}\},\{\textbf{def}\ \ \text{X3} : \text{X4} \to \text{X5}\}) \quad \text{(INCOMPAT/RECV-ARROW)}$$

$$\text{incompat-tp}(\{\textbf{def}\ \ \text{X1} : \text{X2} \to \text{X3}\},\{\textbf{val}\ \ \text{X4} : \text{X5}\}) \quad \text{(INCOMPAT/ARROW-RECV)}$$

$$\text{incompat-tp}(\{\textbf{type}\ \ \text{X1} : \text{X2} \mathinner{..} \text{X3}\},\{\textbf{def}\ \ \text{X4} : \text{X5} \to \text{X6}\}) \quad \text{(INCOMPAT/RECT-ARROW)}$$

$$\frac{\begin{array}{c}\text{tpe}\\ \text{tpe}\end{array}}{\text{type}} \quad \text{(COMPAT-TP)}$$

$$\text{compat-tp}(\{\textbf{def}\ \ \text{N} : \text{X1} \to \text{X2}\},\{\textbf{def}\ \ \text{N} : \text{X3} \to \text{X4}\}) \quad \text{(COMPAT/ARROW-ARROW)}$$

$$\text{compat-tp}(\{\textbf{val}\ \ \text{N} : \text{X1}\},\{\textbf{val}\ \ \text{N} : \text{X2}\}) \quad \text{(COMPAT/RECV-RECV)}$$

$$\text{compat-tp}(\{\textbf{type}\ \ \text{N} : \text{X1} \mathinner{..} \text{X2}\},\{\textbf{type}\ \ \text{N} : \text{X3} \mathinner{..} \text{X4}\}) \quad \text{(COMPAT/RECT-RECT)}$$

$$\frac{\begin{array}{c}\text{incompat-tp(T1,T3)}\\ \text{compat-tp(T1,T3)}\\ \text{false}\end{array}}{\text{type}} \quad \text{(NO-SUBTYPE0)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{E1} : \text{T1}\\ \text{G} \vdash \text{T1}\ \textbf{wf}_!\end{array}}{\text{type}} \quad \text{(EXTRACT-WF2)}$$

$$\text{extract-wf2(t/empty,wf-tp/top)}$$

$$\text{extract-wf2(t/var(W,L),W)}$$

$$\text{extract-wf2(t/fun(X14,WFF,X15,X16,X17,X18,X19,X20),WFF)}$$

$$\frac{\text{extract-wf2(TS2,wf-tp/fun(W2,W1))}}{\text{extract-wf2(t/app(TS1,TS2),W2)}}$$

$$\frac{\text{extract-wf2(TS,wf-tp/recv(W))}}{\text{extract-wf2(t/sel(TS),W)}}$$

$$\frac{\text{extract-wf(S,W1,W2)}}{\text{extract-wf2(t/sub(S,TS),W2)}}$$

$$\frac{\begin{array}{c}\text{V} \vdash \text{G}\ \mathbf{wf}_v \\ \text{G} \vdash \text{T1}\ \mathbf{wf}_{\approx}\end{array}}{\text{type}} \quad \text{(EXTRACT-WF3)}$$

$$\text{extract-wf3(wfv/empty,wf-tp/top)}$$

$$\frac{\text{extract-wf(SFF,X5,WFF)}}{\text{extract-wf3(wfv/f(X17,SFF,X18,X19,X20,X21,X22),WFF)}}$$

$$\text{extract-wf3(wfv/t(WR),WR)}$$

$$\frac{\text{extract-wf(S,W1,W2)}}{\text{extract-wf3(wfv/sub(S,TS),W2)}}$$

$$\frac{\begin{array}{c}\text{X3} \vdash \text{X2}\ \mathbf{wf}_v \\ \text{Z is tpe} \\ \text{X3} \vdash \text{Z::X2}\ \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(EXTEND-WFV)}$$

$$\text{extend-wfv(wfv/empty,T,wfv/empty)}$$

$$\frac{\text{extend-sub-tp2(ST,T,ST1)}}{\text{extend-wfv(wfv/sub(ST,WV),T,wfv/sub(ST1,WV))}}$$

$$\frac{\text{extend-sub-tp2(ST,T,ST1)}}{\text{extend-wfv(wfv/f(IX,ST,MT,TV,TV2,TE,WE),T,wfv/f(IX,ST1,MT,TV,TV2,TE,WE))}}$$

$$\frac{\text{extend-wf-tp(WR,T,WR1)}}{\text{extend-wfv(wfv/t(WR),T,wfv/t(WR1))}}$$

$$\frac{\begin{array}{c}\text{G1} \sqsubseteq \text{G2} \\ \text{V} \vdash \text{G1}\ \mathbf{wf}_v \\ \text{V} \vdash \text{G2}\ \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(WFV-WIDEN-ENV)}$$

$$\text{wfv-widen-env(sub-env/refl,WV,WV)}$$

$$\frac{\begin{array}{c}\text{extend-wfv(WV1,Z,WVX)} \\ \text{wfv-widen-env(S,WV,WV1)}\end{array}}{\text{wfv-widen-env(sub-env/ext(S),WV,WVX)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\approx} \text{T2} \dashv \text{G2} \\ \text{V} \vdash \text{G1}\ \mathbf{wf}_v \\ \text{V} \vdash \text{G2}\ \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(WFV-WIDEN)}$$

$$\text{wfv-widen(S,W,wfv/sub(S,W))}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{E1.LNV} : \text{T} \\ \text{G} \vdash \text{E1} : \{\mathbf{val}\ \ \text{LNV} : \text{T1}\} \\ \text{G} \vdash \text{T1} <:_{\approx} \text{T} \dashv \text{G}\end{array}}{\text{type}} \quad \text{(INVERT-SEL)}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(W',SB)}\\ \text{widen-mode-wf-tp(W,W')}\\ \text{extract-wf2(t/sel(TS),W)}\end{array}}{\text{invert-sel(t/sel(TS),TS,SB)}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(SB,SB1',SB2)}\\ \text{widen-mode-sub-tp(SB1,SB1')}\\ \text{invert-sel(TS,L2,SB)}\end{array}}{\text{invert-sel(t/sub(SB1,TS),L2,SB2)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{var(N)} : \text{T}\\ \text{N} \mapsto \text{T1} \in \text{G}\\ \text{G} \vdash \text{T1} <:_{\approx} \text{T} \dashv \text{G}\end{array}}{\text{type}} \quad \text{(INVERT-VAR)}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(W',ST)}\\ \text{widen-mode-wf-tp(W,W')}\end{array}}{\text{invert-var(t/var(W,L),L,ST)}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(ST,SB1',ST1)}\\ \text{widen-mode-sub-tp(SB1,SB1')}\\ \text{invert-var(TS,P,ST)}\end{array}}{\text{invert-var(t/sub(SB1,TS),P,ST1)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{E1.LNF(E2)} : \text{T}\\ \text{G} \vdash \text{E1} : \{\textbf{def}\ \ \text{LNF} : \text{T1} \to \text{T2}\}\\ \text{G} \vdash \text{E2} : \text{T1}\\ \text{G} \vdash \text{T2} <:_{\approx} \text{T} \dashv \text{G}\end{array}}{\text{type}} \quad \text{(INVERT-APP)}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(W',SB)}\\ \text{widen-mode-wf-tp(W,W')}\\ \text{extract-wf2(t/app(TS1,TS),W)}\end{array}}{\text{invert-app(t/app(TS1,TS),TS,TS1,SB)}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(SB,SB1',SB2)}\\ \text{widen-mode-sub-tp(SB1,SB1')}\\ \text{invert-app(TS,L,L2,SB)}\end{array}}{\text{invert-app(t/sub(SB1,TS),L,L2,SB2)}}$$

$$\frac{\begin{array}{c}\text{G} \vdash \textbf{new}\ \text{TC}\ \{\textbf{def}\ \text{LNF(\_:X2):R=X3;}\textbf{val}\ \text{LNV:R2=X4;}\textbf{types}\ \text{M}\} : \text{T}\\ \text{T3::TC::G} \vdash \text{R} : \text{T4}\\ \text{MT::G} \vdash \text{R2} : \text{T1}\\ \text{type-mem(M,MT)}\\ \text{MT::G} \vdash \text{MT}\ \textbf{wf}_{\approx}\\ \text{MT::G} \vdash \text{T1} <:_{\approx} \text{T1} \dashv \text{TC::G}\\ \text{TC::G} \vdash \text{TC} <:_{\approx} \text{T} \dashv \text{G}\\ \text{TC::G} \vdash \{\textbf{def}\ \ \text{LNF} : \text{T3} \to \text{T4}\} \wedge \{\textbf{val}\ \ \text{LNV} : \text{T1}\} \wedge \text{MT} <:_{\approx} \text{TC} \dashv \text{TC::G}\end{array}}{\text{type}} \quad \text{(INVERT-FUN)}$$

$$\frac{\begin{array}{c}\text{extend-sub-tp(ST,X2,X3,ST',X4)}\\ \text{sub-tp-refl(WFF',ST)}\\ \text{widen-mode-sub-tp(IF,IF')}\\ \text{widen-mode-sub-tp(STV,STV')}\\ \text{widen-mode-wf-tp(WR,WR')}\\ \text{widen-mode-wf-tp(WFF,WFF')}\end{array}}{\text{invert-fun(t/fun(IF,WFF,WR,STV,TV,TS,M,S),TS,TV,M,WR',STV',ST',IF')}}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(ST,SB1',ST')}\\ \text{widen-mode-sub-tp(SB1,SB1')}\\ \text{invert-fun(TS,TSV,L,M,WR,STV,ST,IF)}\end{array}}{\text{invert-fun(t/sub(SB1,TS),TSV,L,M,WR,STV,ST',IF)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \{\textbf{val}\ \ \text{X1} : \text{X2}\} <:_{\text{M}} \{\textbf{def}\ \ \text{X3} : \text{X4} \to \text{X5}\} \dashv \text{G2}\\ \text{false}\end{array}}{\text{type}}\ (\textsc{no-subtype1})$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \text{T2} \dashv \text{G2}\\ \text{incompat-tp(T1,T2)}\\ \text{false}\end{array}}{\text{type}}\ (\textsc{no-subtype2})$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \{\textbf{def}\ \ \text{NF} : \text{TA} \to \text{TB}\} \dashv \text{G2}\\ \text{type-mem(M',T1)}\\ \text{false}\end{array}}{\text{type}}\ (\textsc{no-subtype-mt-arrow})$$

$$\frac{\text{no-subtype-mt-arrow(ST1,TM,F)}}{\text{no-subtype-mt-arrow(sub-tp/and1b(X12,ST1),tm/cons(TM),F)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\text{M}} \{\textbf{val}\ \ \text{NF} : \text{TA}\} \dashv \text{G2}\\ \text{type-mem(M',T1)}\\ \text{false}\end{array}}{\text{type}}\ (\textsc{no-subtype-mt-recv})$$

$$\frac{\text{no-subtype-mt-recv(ST1,TM,F)}}{\text{no-subtype-mt-recv(sub-tp/and1b(X11,ST1),tm/cons(TM),F)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \{\textbf{val}\ \ \text{N1} : \text{T1}\} <:_{\approx} \{\textbf{val}\ \ \text{N2} : \text{T2}\} \dashv \text{G2}\\ \text{N1} = \text{N2}\end{array}}{\text{type}}\ (\textsc{sub-recv-eq})$$

$$\frac{\text{eq-refl(X1,EQ)}}{\text{sub-recv-eq(sub-tp/recv(S),EQ)}}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \{\textbf{def}\ \ \text{NF} : \text{T1} \to \text{T2}\} \wedge \{\textbf{val}\ \ \text{NV'} : \text{T3}\} \wedge \text{MT} <:_{\approx} \{\textbf{val}\ \ \text{NV} : \text{TX}\} \dashv \text{G2}\\ \text{type-mem(M',MT)}\\ \text{G1} \vdash \{\textbf{val}\ \ \text{NV'} : \text{T3}\} <:_{\approx} \{\textbf{val}\ \ \text{NV} : \text{TX}\} \dashv \text{G2}\end{array}}{\text{type}}\ (\textsc{project-sub-recv})$$

$$\frac{\begin{array}{c}\text{no-subtype(F,}\approx\text{,X1,\{}\textbf{val}\ \ \text{X2} : \text{X3\},X4,\{}\textbf{val}\ \ \text{X5} : \text{X6\},ST')}\\ \text{no-subtype2(ST,incompat/arrow-recv,F)}\end{array}}{\text{project-sub-recv(sub-tp/and1a(W,ST),MT,ST')}}$$

$$\frac{\text{project-sub-recv(sub-tp/and1b(WF,sub-tp/and1a(WR,ST)),MT,ST)}\quad \text{no-subtype(F,}\approx\text{,X1,}\{\textbf{val}\ \ \text{X2} : \text{X3}\}\text{,X4,}\{\textbf{val}\ \ \text{X5} : \text{X6}\}\text{,ST')}\quad \text{no-subtype-mt-recv(ST,MT,F)}}{\text{project-sub-recv(sub-tp/and1b(WF,sub-tp/and1b(WV,ST)),MT,ST')}}$$

$$\frac{\text{G1} \vdash \{\textbf{def}\ \ \text{N1} : \text{T1A} \to \text{T1B}\} <:_{\approx} \{\textbf{def}\ \ \text{N2} : \text{T2A} \to \text{T2B}\} \dashv \text{G2}\quad \text{N1} = \text{N2}}{\text{type}}\ (\textsc{sub-arrow-eq})$$

$$\frac{\text{eq-refl(X1,EQ)}}{\text{sub-arrow-eq(sub-tp/fun(X8,X9),EQ)}}$$

$$\frac{\text{G1} \vdash \{\textbf{def}\ \ \text{NF'} : \text{T1} \to \text{T2}\} \wedge \{\textbf{val}\ \ \text{NV} : \text{T3}\} \wedge \text{MT} <:_{\approx} \{\textbf{def}\ \ \text{NF} : \text{T1X} \to \text{T2X}\} \dashv \text{G2}\quad \text{type-mem(M',MT)}\quad \text{G1} \vdash \{\textbf{def}\ \ \text{NF'} : \text{T1} \to \text{T2}\} <:_{\approx} \{\textbf{def}\ \ \text{NF} : \text{T1X} \to \text{T2X}\} \dashv \text{G2}}{\text{type}}\ (\textsc{project-sub-arrow})$$

$$\frac{\text{project-sub-arrow(sub-tp/and1a(W,ST),MT,ST)}\quad \text{no-subtype(F,}\approx\text{,X1,}\{\textbf{def}\ \ \text{X2} : \text{X3} \to \text{X4}\}\text{,X5,}\{\textbf{def}\ \ \text{X6} : \text{X7} \to \text{X8}\}\text{,ST')}\quad \text{no-subtype2(ST,incompat/recv-arrow,F)}}{\text{project-sub-arrow(sub-tp/and1b(WF,sub-tp/and1a(WR,ST)),MT,ST')}}$$

$$\frac{\text{project-sub-arrow(sub-tp/and1a(W,ST),MT,ST)}\quad \text{no-subtype(F,}\approx\text{,X1,}\{\textbf{def}\ \ \text{X2} : \text{X3} \to \text{X4}\}\text{,X5,}\{\textbf{def}\ \ \text{X6} : \text{X7} \to \text{X8}\}\text{,ST')}\quad \text{no-subtype-mt-arrow(ST,MT,F)}}{\text{project-sub-arrow(sub-tp/and1b(WF,sub-tp/and1b(WV,ST)),MT,ST')}}$$

$$\frac{< \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G1}\ \textbf{wf}_v\quad \text{G1} \vdash \text{T} <:_{\approx} \{\textbf{def}\ \ \text{LNF} : \text{T1} \to \text{T2}\} \dashv \text{G2}\quad \text{T3::TS::G} \vdash \text{R} : \text{T4}\quad \text{wf-env(H,G)}\quad \text{TS::G} \vdash \{\textbf{def}\ \ \text{LNF} : \text{T3} \to \text{T4}\} <:_{\approx} \{\textbf{def}\ \ \text{LNF} : \text{T1} \to \text{T2}\} \dashv \text{G2}\quad < \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{TS::G}\ \textbf{wf}_v\quad \text{LNF'} = \text{LNF}}{\text{type}}\ (\textsc{invert-wf-fun-aux})$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(WFF',ST')}\\ \text{extract-wf(ST,WFF',X5)}\\ \text{eq-arrow-low(EQ,STF1,STF1')}\\ \text{sub-arrow-eq(STF1,EQ)}\\ \text{project-sub-arrow(SB0",MT,STF1)}\\ \text{sub-tp-trans(IN,SB0,SB0")}\\ \text{sub-tp-trans(ST,SB0',SB0)}\end{array}}{\text{invert-wf-fun-aux(wfv/f(IN,ST,MT,TV,TV2,TS,WE),SB0',TS,WE,STF1',wfv/f(IN,ST',MT,TV,TV2,TS,WE),EQ)}}$$

$$\frac{\text{invert-wf-fun-aux(TS,SB1X,L,L1,SB,WVXX,EQ)}\quad \text{sub-tp-trans(SB1,SB0,SB1X)}}{\text{invert-wf-fun-aux(wfv/sub(SB1,TS),SB0,L,L1,SB,WVXX,EQ)}}$$

$$\frac{< \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{G1}\ \textbf{wf}_v\quad \text{T3::TS::G} \vdash \text{R} : \text{T4}\quad \text{wf-env(H,G)}\quad \text{TS::G} \vdash \{\textbf{def}\ \ \text{LNF} : \text{T3} \to \text{T4}\} <:_{\approx} \{\textbf{def}\ \ \text{LNF} : \text{T1} \to \text{T2}\} \dashv \text{G1}\quad < \{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash \text{TS::G}\ \textbf{wf}_v\quad \text{LNF'} = \text{LNF}}{\text{type}}\ (\textsc{invert-wf-fun})$$

$$\frac{\begin{array}{c}\text{invert-wf-fun-aux(A,SB,B,C,D,E,EQ)}\\ \text{sub-tp-refl(W,SB)}\\ \text{extract-wf3(A,W)}\end{array}}{\text{invert-wf-fun(A,B,C,D,E,EQ)}}$$

$$\frac{\begin{array}{c}< \{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV'} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash\ \text{G1}\ \textbf{wf}_v\\ \text{G1} \vdash \text{T} <:_{\approx} \{\textbf{val}\ \ \text{LNV} : \text{T2}\} \dashv \text{G2}\\ \text{R2} \vdash\ \text{TS::G}\ \textbf{wf}_v\\ \text{TS::G} \vdash \{\textbf{val}\ \ \text{LNV} : \text{T4}\} <:_{\approx} \{\textbf{val}\ \ \text{LNV} : \text{T2}\} \dashv \text{G2}\\ \text{LNV'} = \text{LNV}\end{array}}{\text{type}}\ (\textsc{invert-wf-rec-aux})$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(sub-tp/recv(STV),SB1'',SB1)}\\ \text{eq-recv-low(EQ,SB1',SB1'')}\\ \text{sub-recv-eq(SB1',EQ)}\\ \text{project-sub-recv(SB0'',MT,SB1')}\\ \text{sub-tp-trans(IN,SB0,SB0'')}\\ \text{sub-tp-trans(ST,SB0',SB0)}\end{array}}{\text{invert-wf-rec-aux(wfv/f(IN,ST,MT,TV,STV,TS,WE),SB0',TV,SB1,EQ)}}$$

$$\frac{\begin{array}{c}\text{invert-wf-rec-aux(TS,SB1X,TV,SB,EQ)}\\ \text{sub-tp-trans(SB1,SB0,SB1X)}\end{array}}{\text{invert-wf-rec-aux(wfv/sub(SB1,TS),SB0,TV,SB,EQ)}}$$

$$\frac{\begin{array}{c}< \{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV'} = \text{R2}\}\ \textbf{in}\ \ \text{H}>_v \vdash\ \text{G1}\ \textbf{wf}_v\\ \text{R2} \vdash\ \text{G}\ \textbf{wf}_v\\ \text{G} \vdash \{\textbf{val}\ \ \text{LNV} : \text{T4}\} <:_{\approx} \{\textbf{val}\ \ \text{LNV} : \text{T}\} \dashv \text{G1}\\ \text{LNV'} = \text{LNV}\end{array}}{\text{type}}\ (\textsc{invert-wf-rec})$$

$$\frac{\begin{array}{c}\text{invert-wf-rec-aux(A,SB,B,C,EQ)}\\ \text{sub-tp-refl(W,SB)}\\ \text{extract-wf3(A,W)}\end{array}}{\text{invert-wf-rec(A,B,C,EQ)}}$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{tlookup-zero(G,N1,T)}\\ \text{vlookup-zero(H,N1,V)}\\ \text{V} \vdash\ \text{G}\ \textbf{wf}_v\end{array}}{\text{type}}\ (\textsc{lookup-zero-safe-total})$$

$$\text{lookup-zero-safe-total(wfe/c(G,V),tl/hit,vl/hit,V)}$$

$$\frac{\begin{array}{c}\text{extend-wfv(Z,X4,Z1)}\\ \text{lookup-zero-safe-total(G,A,B,Z)}\end{array}}{\text{lookup-zero-safe-total(wfe/c(G,V),tl/miss(A),vl/miss(B),Z1)}}$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{tlookup-zero(G,N2,T)}\\ \text{vlookup-zero(H,N1,V)}\\ \text{N1} = \text{N2}\\ \text{V} \vdash\ \text{G}\ \textbf{wf}_v\end{array}}{\text{type}}\ (\textsc{lookup-zero-safe})$$

$$\text{lookup-zero-safe(wfe/c(G,V),tl/hit,vl/hit,eq/z,V)}$$

$$\frac{\text{extend-wfv(Z,X4,Z1)} \quad \text{lookup-zero-safe(G,A,B,E,Z)}}{\text{lookup-zero-safe(wfe/c(G,V),tl/miss(A),vl/miss(B),eq/s(E),Z1)}}$$

$$\frac{\text{wf-env(H,G)} \quad |H|{=}N1 \quad |G|{=}N2 \quad N1 = N2}{\text{type}} \text{ (WF-ENV-SIZE-EQ)}$$

$$\text{wf-env-size-eq(wfe/n,vf/n,tf/n,eq/z)}$$

$$\frac{\text{wf-env-size-eq(G,VS,TS,E)}}{\text{wf-env-size-eq(wfe/c(G,X7),vf/c(VS),tf/c(TS),eq/s(E))}}$$

$$\frac{\text{wf-env(H,G)} \quad N \mapsto T \in G \quad N \mapsto V \in H \quad V \vdash G \ \mathbf{wf}_v}{\text{type}} \text{ (LOOKUP-SAFE)}$$

$$\frac{\text{lookup-zero-safe(WE,TL,VL,EM,WV)} \quad \text{sub-eq(EN,ES,VA,TA,EM)} \quad \text{wf-env-size-eq(WE,VS,TS,ES)} \quad \text{eq-refl(s(N),EN)}}{\text{lookup-safe(WE,tl(TL,TA,TS),vl(VL,VA,VS),WV)}}$$

$$\frac{\text{wf-env(H,G)} \quad |G|{=}N \quad |H|{=}N}{\text{type}} \text{ (WF-ENV-TSIZE-VSIZE)}$$

$$\text{wf-env-tsize-vsize(wfe/n,tf/n,vf/n)}$$

$$\frac{\text{wf-env-tsize-vsize(G,TS,VS)}}{\text{wf-env-tsize-vsize(wfe/c(G,X6),tf/c(TS),vf/c(VS))}}$$

$$\frac{\text{wf-env(H,G)} \quad N \mapsto T \in G \quad N \mapsto V \in H \quad V \vdash G \ \mathbf{wf}_v}{\text{type}} \text{ (LOOKUP-SAFE-TOTAL)}$$

$$\frac{\text{lookup-zero-safe-total(WE,TL,VL,WV)} \quad \text{wf-env-tsize-vsize(WE,TS,VS)}}{\text{lookup-safe-total(WE,tl(TL,A,TS),vl(VL,A,VS),WV)}}$$

$$\frac{G1 \vdash \{\mathbf{val} \ \ \text{LNV} : T1\} <:_{\approx} \{\mathbf{val} \ \ \text{LNV} : T2\} \dashv G2 \quad G1 \vdash T1 <:_{\approx} T2 \dashv G2}{\text{type}} \text{ (SUB-TP-INV-RECV)}$$

$$\text{sub-tp-inv-recv(sub-tp/recv(S),S)}$$

$$\frac{\begin{array}{c}\mathrm{G1} \vdash \{\mathbf{def}\ \ \mathrm{LNF} : \mathrm{T1} \to \mathrm{T2}\} <:_{\approx} \{\mathbf{def}\ \ \mathrm{LNF} : \mathrm{T3} \to \mathrm{T4}\} \dashv \mathrm{G2} \\ \mathrm{G2} \vdash \mathrm{T3} <:_{\approx} \mathrm{T1} \dashv \mathrm{G1} \\ \mathrm{G1} \vdash \mathrm{T2} <:_{\approx} \mathrm{T4} \dashv \mathrm{G2}\end{array}}{\mathrm{type}} \quad (\textsc{sub-tp-inv-fun})$$

sub-tp-inv-fun(sub-tp/fun(S1,S2),S2,S1)

$$\frac{\begin{array}{c}\mathrm{G} \vdash \{\emptyset_e\} : \mathrm{T} \\ \{\emptyset_v\} \vdash \mathrm{G}\ \mathbf{wf}_v\end{array}}{\mathrm{type}} \quad (\textsc{eval-empty-safe})$$

eval-empty-safe(t/empty,wfv/empty)

$$\frac{\begin{array}{c}\text{eval-empty-safe(ZT,W)} \\ \text{widen-mode-sub-tp(ST,ST')}\end{array}}{\text{eval-empty-safe(t/sub(ST,ZT),wfv/sub(ST',W))}}$$

$$\frac{\begin{array}{c}\mathrm{G} \vdash \mathrm{E} : \mathrm{T} \\ \text{wf-env(H,G)} \\ \mathrm{H} \vdash \mathrm{E} \Downarrow \mathrm{V} \\ \mathrm{V} \vdash \mathrm{G}\ \mathbf{wf}_v\end{array}}{\mathrm{type}} \quad (\textsc{eval-safe})$$

$$\frac{\text{eval-empty-safe(ZT,V)}}{\text{eval-safe(ZT,X4,e/empty,V)}}$$

$$\frac{\begin{array}{c}\text{wfv-widen(ST,V,V2)} \\ \text{lookup-safe(E,L,L2,V)} \\ \text{invert-var(ZT,L,ST)}\end{array}}{\text{eval-safe(ZT,E,e/var(L2),V2)}}$$

$$\frac{\begin{array}{c}\text{sub-tp-refl(WR,STM)} \\ \text{sub-tp-refl(WFF,STFF)} \\ \text{extract-wf(CST,WFF,X10)} \\ \text{invert-fun(ZT,TS,TSV,M,WR,STV,CST,IN)}\end{array}}{\begin{array}{r}\text{eval-safe(TSV,wfe/c(E,wfv/f(sub-tp/and1b(wf-tp/fun(wf-tp/top,wf-tp/bot),sub-tp/and1b(wf-tp/recv(wf-tp/top),STM)),STM,} \\ \text{eval-safe(ZT,E,e/fun(EX),wfv/sub(CST,wfv/f(IN,STFF,M,TV,STV,TS,}\end{array}}$$

$$\frac{\begin{array}{c}\text{wfv-widen(ST,WV2,WV3)} \\ \text{wfv-widen(ST1,WV1,WV2)} \\ \text{sub-tp-inv-recv(STR,ST1)} \\ \text{invert-wf-rec(WVX,WV1,STR,X11)} \\ \text{eval-safe(TS,WE,EVX,WVX)} \\ \text{invert-sel(ZT,TS,ST)}\end{array}}{\text{eval-safe(ZT,WE,e/sel(EVX),WV3)}}$$

$$\frac{\begin{array}{c}\text{wfv-widen(ST,Z1,Z2)}\\ \text{wfv-widen(STR',Z,Z1)}\\ \text{extend-sub-tp1(STR,T3,STR')}\\ \text{eval-safe(TR,wfe/c(wfe/c(WE1,WXX),WVX3),EV3,Z)}\\ \text{wfv-widen(STA',WVX,WVX3)}\\ \text{extend-sub-tp2(STA,T3,STA')}\\ \text{sub-tp-inv-fun(STF,STA,STR)}\\ \text{invert-wf-fun(ZF,TR,WE1,STF,WXX,X4)}\\ \text{eval-safe(TF,WE,EVF,ZF)}\\ \text{eval-safe(TX,WE,EVX,WVX)}\\ \text{invert-app(ZT,TF,TX,ST)}\end{array}}{\text{eval-safe(ZT,WE,e/app(EV3,EVX,EVF),Z2)}}$$

$$\frac{\text{type}}{}\quad\text{(RES-VAL)}$$

$$\frac{\text{res-val}}{}\quad\text{(RES-VAL/STUCK)}$$

$$\frac{\text{res-val}}{}\quad\text{(RES-VAL/TIMEOUT)}$$

$$\frac{\text{val}}{\text{res-val}}\quad\text{(RES-VAL/SOME)}$$

$$\frac{\begin{array}{c}\text{venv}\\ \text{nat}\\ \text{res-val}\end{array}}{\text{type}}\quad\text{(VLOOKUP-ZERO-RES)}$$

$$\frac{\text{vlookup-zero-res(}\emptyset\text{,N,res-val/stuck)}}{}\quad\text{(VLR/FAIL)}$$

$$\frac{\text{vlookup-zero-res(V::G,z,res-val/some(V))}}{}\quad\text{(VLR/HIT)}$$

$$\frac{\text{vlookup-zero-res(G,N,OV)}}{\text{vlookup-zero-res(X1::G,s(N),OV)}}\quad\text{(VLR/MISS)}$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\\ \text{nat}\end{array}}{\text{type}}\quad\text{(RES-ADD)}$$

$$\frac{\text{res-add(N1,N2,N3)}}{}\quad\text{(RES-ADD/STUCK)}$$

$$\frac{\text{N1 + N2 = N3}}{\text{res-add(N1,N2,N3)}}\quad\text{(RES-ADD/SOME)}$$

$$\frac{\begin{array}{c}\text{res-add(N1,N2,N3)}\\ \text{res-add(s(N1),N2,s(N3))}\end{array}}{\text{type}}\quad\text{(RES-ADD-INC)}$$

$$\frac{\text{res-add-inc(res-add/stuck,res-add/stuck)}}{}\quad\text{(RES-ADD-INC/STUCK)}$$

$$\frac{\text{res-add-inc(res-add/some(X),res-add/some(add/s(X)))}}{}\quad\text{(RES-ADD-INC/SOME)}$$

$$\frac{\begin{array}{c}\text{N3 is nat}\\ \text{N1 is nat}\\ \text{N2 is nat}\\ \text{res-add(N1,N2,N3)}\end{array}}{\text{type}}\quad\text{(MINUS)}$$

$$\text{minus(z,s(N1),z,res-add/stuck)} \qquad \text{(MINUS/STUCK)}$$

$$\text{minus(N,z,N,res-add/some(add/z))} \qquad \text{(MINUS/Z)}$$

$$\frac{\begin{array}{c}\text{res-add-inc(R,R')}\\ \text{minus(N3,N1,N2,R)}\end{array}}{\text{minus(s(N3),s(N1),N2,R')}} \qquad \text{(MINUS/S)}$$

$$\frac{\begin{array}{c}\text{venv}\\ \text{N2 is nat}\\ \text{res-add(N1,N2,N3)}\\ \text{res-val}\end{array}}{\text{type}} \qquad \text{(VLOOKUP-ZERO-RES2)}$$

$$\text{vlookup-zero-res2(E,N2,res-add/stuck,res-val/stuck)} \qquad \text{(VLR2/STUCK)}$$

$$\frac{\text{vlookup-zero-res(E,N2,OV)}}{\text{vlookup-zero-res2(E,N2,res-add/some(X3),OV)}} \qquad \text{(VLR2/SOME)}$$

$$\frac{\begin{array}{c}\text{venv}\\ \text{nat}\\ \text{res-val}\end{array}}{\text{type}} \qquad \text{(VLOOKUP-RES)}$$

$$\frac{\begin{array}{c}\text{vlookup-zero-res2(G,M,R,V)}\\ \text{minus(S,s(N),M,R)}\\ |\text{G}|=\text{S}\end{array}}{\text{vlookup-res(G,N,V)}} \qquad \text{(VLR)}$$

$$\frac{\begin{array}{c}\text{res-val}\\ \text{nat}\\ \text{exp}\\ \text{nat}\\ \text{venv}\\ \text{res-val}\end{array}}{\text{type}} \qquad \text{(RES-CLOS)}$$

$$\text{res-clos(res-val/stuck,X1,X2,X3,X4,res-val/stuck)} \qquad \text{(RES-CLOS/STUCK)}$$

$$\text{res-clos(res-val/timeout,X1,X2,X3,X4,res-val/timeout)} \qquad \text{(RES-CLOS/TIMEOUT)}$$

$$\text{res-clos(res-val/some(V2),LNF,R,LNV,G,res-val/some}(<\{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{V2}\}\ \textbf{in}\ \ \text{G}>_v)) \qquad \text{(RES-CLOS/SOME)}$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\\ \text{res-val}\\ \text{res-val}\\ \text{res-val}\end{array}}{\text{type}} \qquad \text{(IF-EQ-THEN-ELSE)}$$

$$\text{if-eq-then-else(z,z,A,B,A)} \qquad \text{(IFR/ZZ)}$$

$$\frac{\text{if-eq-then-else(N1,N2,A,B,C)}}{\text{if-eq-then-else(s(N1),s(N2),A,B,C)}} \qquad \text{(IFR/SS)}$$

$$\text{if-eq-then-else(s(N1),z,A,B,B)} \qquad (\text{IFR/SZ})$$

$$\text{if-eq-then-else(z,s(N2),A,B,B)} \qquad (\text{IFR/ZS})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{res-val}\\ \text{res-val}\end{array}}{\text{type}} \qquad (\text{EVAL-SEL-RES})$$

$$\frac{\text{if-eq-then-else(LNV,LNV',res-val/some(V),res-val/stuck,OV)}}{\text{eval-sel-res(LNV,res-val/some(}<\{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV'} = \text{V}\}\ \textbf{in}\ \ \text{G1}>_v\text{),OV)}} \qquad (\text{ES/OK})$$

$$\text{eval-sel-res(LNV,res-val/some(}\{\emptyset_v\}\text{),res-val/stuck)} \qquad (\text{ES/EMPTYFAIL})$$

$$\text{eval-sel-res(LNV,res-val/stuck,res-val/stuck)} \qquad (\text{ES/STUCK})$$

$$\text{eval-sel-res(LNV,res-val/timeout,res-val/timeout)} \qquad (\text{ES/TIMEOUT})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{venv}\\ \text{exp}\\ \text{res-val}\end{array}}{\text{type}} \qquad (\text{EVAL-EXP-RES})$$

$$\frac{\begin{array}{c}\text{nat}\\ \text{nat}\\ \text{res-val}\\ \text{res-val}\\ \text{res-val}\end{array}}{\text{type}} \qquad (\text{EVAL-APP-RES})$$

$$\text{eval-app-res(z,LNF,OV1,OV2,res-val/timeout)} \qquad (\text{EA/TIMEOUT})$$

$$\frac{\begin{array}{c}\text{if-eq-then-else(LNF,LNF',OV3,res-val/stuck,OV3')}\\ \text{eval-exp-res(N,V2::}<\{\textbf{def}\ \text{LNF} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{G1}>_v\text{::G1,R,OV3)}\end{array}}{\text{eval-app-res(s(N),LNF,res-val/some(}<\{\textbf{def}\ \text{LNF'} = \text{R};\ \textbf{val}\ \ \text{LNV} = \text{R2}\}\ \textbf{in}\ \ \text{G1}>_v\text{),res-val/some(V2),OV3')}} \qquad (\text{EA/OK})$$

$$\text{eval-app-res(s(N),LNF,res-val/some(}\{\emptyset_v\}\text{),OV2,res-val/stuck)} \qquad (\text{EA/EMPTYFAIL})$$

$$\text{eval-app-res(N,LNF,res-val/stuck,OV,res-val/stuck)} \qquad (\text{EA/STUCK1})$$

$$\text{eval-app-res(N,LNF,OV,res-val/stuck,res-val/stuck)} \qquad (\text{EA/STUCK2})$$

$$\text{eval-app-res(N,LNF,res-val/timeout,OV,res-val/timeout)} \qquad (\text{EA/TIMEOUT1})$$

$$\text{eval-app-res(N,LNF,OV,res-val/timeout,res-val/timeout)} \qquad (\text{EA/TIMEOUT2})$$

$$\text{eval-exp-res(z,G,E,res-val/timeout)} \qquad (\text{ER/TIMEOUT})$$

$$\text{eval-exp-res(X1,G,}\{\emptyset_e\}\text{,res-val/some(}\{\emptyset_v\}\text{))} \qquad (\text{ER/EMPTY})$$

$$\frac{\text{vlookup-res(G,N,V)}}{\text{eval-exp-res(X1,G,var(N),V)}} \qquad (\text{ER/VAR})$$

$$\frac{\begin{array}{c}\text{res-clos(OV2,LNF,R,LNV,G,C)}\\ \text{eval-exp-res(N,<}\{\textbf{def}\ \text{z} = \{\emptyset_e\};\ \textbf{val}\ \ \text{z} = \{\emptyset_v\}\}\ \textbf{in}\ \ \text{G}>_v\text{::G,R2,OV2)}\end{array}}{\text{eval-exp-res(s(N),G,}\textbf{new}\ \text{TC}\ \{\textbf{def}\ \text{LNF(\_:X2):R=X3;}\textbf{val}\ \text{LNV:R2=X4;}\textbf{types}\ \text{MT}\}\text{,C)}} \quad \text{(ER/FUN)}$$

$$\frac{\begin{array}{c}\text{eval-app-res(N,LNF,OV1,OV2,OV3)}\\ \text{eval-exp-res(N,G,E2,OV2)}\\ \text{eval-exp-res(N,G,E1,OV1)}\end{array}}{\text{eval-exp-res(s(N),G,E1.LNF(E2),OV3)}} \quad \text{(ER/APP)}$$

$$\frac{\begin{array}{c}\text{eval-sel-res(LNV,OV1,OV2)}\\ \text{eval-exp-res(N,G,E1,OV1)}\end{array}}{\text{eval-exp-res(s(N),G,E1.LNV,OV2)}} \quad \text{(ER/SEL)}$$

$$\frac{\begin{array}{c}\text{res-val}\\ \text{val}\end{array}}{\text{type}} \quad \text{(RES-VAL-GET)}$$

$$\text{res-val-get(res-val/some(V),V)} \quad \text{(RES-VAL-GET/SOME)}$$

$$\frac{\begin{array}{c}\text{res-val}\\ \text{tenv}\\ \text{tpe}\end{array}}{\text{type}} \quad \text{(WF-RES)}$$

$$\frac{\text{V} \vdash \text{G}\ \textbf{wf}_v}{\text{wf-res(res-val/some(V),G,T)}} \quad \text{(WF-RES/SOME)}$$

$$\text{wf-res(res-val/timeout,G,T)} \quad \text{(WF-RES/TIMEOUT)}$$

$$\frac{\begin{array}{c}\text{res-val}\\ \text{res-val}\end{array}}{\text{type}} \quad \text{(RES-VAL-EQ)}$$

$$\text{res-val-eq(A,A)} \quad \text{(RES-VAL-EQ/ID)}$$

$$\frac{\begin{array}{c}\text{res-val-eq(res-val/some(V),OV')}\\ \text{V} \vdash \text{G}\ \textbf{wf}_v\\ \text{wf-res(OV',G,T)}\end{array}}{\text{type}} \quad \text{(EQ-RES-VAL-SOME-WFV)}$$

$$\text{eq-res-val-some-wfv(res-val-eq/id,B,wf-res/some(B))}$$

$$\frac{\begin{array}{c}\text{res-val-eq(A,B)}\\ \text{wf-res(A,G,T)}\\ \text{wf-res(B,G,T)}\end{array}}{\text{type}} \quad \text{(EQ-WF-RES)}$$

$$\text{eq-wf-res(res-val-eq/id,B,B)}$$

$$\frac{\begin{array}{c}\text{N1 is nat}\\ \text{N2 is nat}\\ \text{N3 is nat}\\ \text{N1 + N2 = N3}\\ \text{N2 + N1 = N3}\end{array}}{\text{type}} \quad \text{(ADD-COMMUTE)}$$

$$\frac{\text{add-commute(z,M,M,add/z,D)}}{\text{add-commute(z,s(M),s(M),add/z,add/s(D))}}$$

$$\text{add-commute(X1,z,X1,X2,add/z)}$$

$$\frac{\begin{array}{c}\text{add-inc(D',D")}\\ \text{add-commute(N1,N2,X1,D,D')}\end{array}}{\text{add-commute(s(N1),N2,s(X1),add/s(D),D")}}$$

$$\frac{\begin{array}{c}\text{C = C'}\\ \text{A + B = C}\\ \text{A + B = C'}\end{array}}{\text{type}} \quad \text{(ADD-EQ)}$$

$$\text{add-eq(eq/z,add/z,add/z)}$$

$$\frac{\begin{array}{c}\text{add-inc(D,C)}\\ \text{add-eq(A,add/z,D)}\end{array}}{\text{add-eq(eq/s(A),add/z,C)}}$$

$$\frac{\text{add-eq(A,B,C)}}{\text{add-eq(eq/s(A),add/s(B),add/s(C))}}$$

$$\frac{\begin{array}{c}\text{natid(B,B')}\\ \text{A + B = C}\\ \text{A + B' = C}\end{array}}{\text{type}} \quad \text{(ADD-NATID2)}$$

$$\text{add-natid2(natident,B,B)}$$

$$\frac{\begin{array}{c}\text{B = B'}\\ \text{A + B = C}\\ \text{A + B' = C}\end{array}}{\text{type}} \quad \text{(ADD-EQ2)}$$

$$\frac{\begin{array}{c}\text{add-natid2(ID,B,B')}\\ \text{eq-to-id(EQ,ID)}\end{array}}{\text{add-eq2(EQ,B,B')}}$$

$$\frac{\begin{array}{c}\text{s(N1) + N2 = z}\\ \text{false}\end{array}}{\text{type}} \quad \text{(ADD-UP-TO-Z-FALSE)}$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{tlookup-zero(G,X,T)}\\ \text{vlookup-zero-res(H,X,OV)}\\ \text{res-val-get(OV,V)}\\ \text{V} \vdash \text{G } \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(LOOKUP-ZERO-SAFE-RES)}$$

$$\text{lookup-zero-safe-res(wfe/c(G,V),tl/hit,vlr/hit,res-val-get/some,V)}$$

$$\frac{\begin{array}{c}\text{extend-wfv(Z,X4,Z')}\\ \text{lookup-zero-safe-res(G,A,B,Y,Z)}\end{array}}{\text{lookup-zero-safe-res(wfe/c(G,V),tl/miss(A),vlr/miss(B),Y,Z')}}$$

$$\frac{\begin{array}{c}\text{false}\\ \text{OV is res-val}\\ \text{V is val}\\ \text{res-val-get(OV,V)}\end{array}}{\text{type}} \quad \text{(NO-RES-VAL-GET)}$$

$$\frac{\begin{array}{c}\text{false}\\ \text{V is val}\\ \text{G is tenv}\\ \text{T is tpe}\\ \text{V} \vdash \text{G } \mathbf{wf}_v\end{array}}{\text{type}} \quad \text{(NO-WFV)}$$

$$\frac{\begin{array}{c}\text{false}\\ \text{OV is res-val}\\ \text{G is tenv}\\ \text{T is tpe}\\ \text{wf-res(OV,G,T)}\end{array}}{\text{type}} \quad \text{(NO-WF-RES)}$$

$$\frac{\begin{array}{c}\text{X + MG = S}\\ \text{minus(S,X,MH,R)}\\ \text{X + MH = S}\\ \text{MG = MH}\end{array}}{\text{type}} \quad \text{(MINUS-TO-ADD)}$$

$$\frac{\text{eq-refl(X1,EQ)}}{\text{minus-to-add(add/z,minus/z,add/z,EQ)}}$$

$$\frac{\text{minus-to-add(A,B2,Y,Z)}}{\text{minus-to-add(add/s(A),minus/s(B1,B2),add/s(Y),Z)}}$$

$$\frac{\begin{array}{c}\text{S = S'}\\ \text{|G|=S}\\ \text{|G|=S'}\end{array}}{\text{type}} \quad \text{(EQ-TSIZE)}$$

$$\text{eq-tsize(eq/z,tf/n,tf/n)}$$

$$\frac{\text{eq-tsize(A,B,C)}}{\text{eq-tsize(eq/s(A),tf/c(B),tf/c(C))}}$$

$$\frac{\begin{array}{c}\text{M = M'}\\ \text{tlookup-zero(G,M,T)}\\ \text{tlookup-zero(G,M',T)}\end{array}}{\text{type}} \quad \text{(EQ-TLOOKUP-ZERO)}$$

$$\text{eq-tlookup-zero(eq/z,tl/hit,tl/hit)}$$

$$\frac{\text{eq-tlookup-zero(A,B,C)}}{\text{eq-tlookup-zero(eq/s(A),tl/miss(B),tl/miss(C))}}$$

$$\frac{\begin{array}{c}\text{s(X) + M = S}\\ \text{minus(S,s(X),M,res-add/stuck)}\\ \text{false}\end{array}}{\text{type}} \quad \text{(ADD-MINUS-CONTRA)}$$

$$\frac{\text{add-up-to-z-false(TA,CONTRA)}}{\text{add-minus-contra(TA,minus/stuck,CONTRA)}}$$

$$\frac{\text{add-minus-contra(TA,B,CONTRA)}}{\text{add-minus-contra(add/s(TA),minus/s(A,B),CONTRA)}}$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{|G|=S}\\ \text{|H|=S}\\ \text{tlookup-zero(G,M,T)}\\ \text{s(X) + M = S}\\ \text{minus(S,s(X),M,R)}\\ \text{vlookup-zero-res2(H,M,R,OV)}\\ \text{res-val-get(OV,V)}\\ \text{V} \vdash \text{G } \mathbf{wf}_v\end{array}}{\text{type}} \quad (\textsc{lookup-zero-safe-res2})$$

$$\frac{\begin{array}{c}\text{no-wfv(CONTRA,V,G,T,WV)}\\ \text{no-res-val-get(CONTRA,res-val/stuck,V,GV)}\\ \text{add-minus-contra(TA,M,CONTRA)}\end{array}}{\text{lookup-zero-safe-res2(WE,TS,VS,TL,TA,M,vlr2/stuck,GV,WV)}}$$

$$\frac{\text{lookup-zero-safe-res(WE,TL,VL,GV,WV)}}{\text{lookup-zero-safe-res2(WE,TS,VS,TL,TA,M,vlr2/some(VL),GV,WV)}}$$

$$\frac{\begin{array}{c}\text{wf-env(H,G)}\\ \text{X} \mapsto \text{T} \in \text{G}\\ \text{vlookup-res(H,X,OV)}\\ \text{res-val-get(OV,V)}\\ \text{V} \vdash \text{G } \mathbf{wf}_v\end{array}}{\text{type}} \quad (\textsc{lookup-safe-res})$$

$$\frac{\begin{array}{c}\text{lookup-zero-safe-res2(WE,TS',VS,TL',TA'',VA,VL,GV,WV)}\\ \text{eq-tsize(ES',TS,TS')}\\ \text{eq-tlookup-zero(EQA,TL,TL')}\\ \text{add-eq2(EQA,TA',TA'')}\\ \text{minus-to-add(TA',VA,VA',EQA)}\\ \text{add-eq(ES',TA,TA')}\\ \text{eq-sym(ES,ES')}\\ \text{wf-env-size-eq(WE,VS,TS,ES)}\end{array}}{\text{lookup-safe-res(WE,tl(TL,TA,TS),vlr(VL,VA,VS),GV,WV)}}$$

$$\frac{\begin{array}{c}\text{res-val-get(OV,V)}\\ \text{V} \vdash \text{G } \mathbf{wf}_v\\ \text{wf-res(OV,G,T)}\end{array}}{\text{type}} \quad (\textsc{to-case-some})$$

$$\text{to-case-some(res-val-get/some,W,wf-res/some(W))}$$

$$\frac{\begin{array}{c}\text{G1} \vdash \text{T1} <:_{\approx} \text{T2} \dashv \text{G2}\\ \text{wf-res(OV,G1,T1)}\\ \text{wf-res(OV,G2,T2)}\end{array}}{\text{type}} \quad (\textsc{wf-res-widen})$$

$$\frac{\text{wfv-widen(S,W,W')}}{\text{wf-res-widen(S,wf-res/some(W),wf-res/some(W'))}}$$

$$\text{wf-res-widen(S,wf-res/timeout,wf-res/timeout)}$$

$$\frac{\begin{array}{c}\text{natid(C,C')}\\ \text{if-eq-then-else(C,C',OA,OB,OC)}\\ \text{if-eq-then-else(C,C',OA,OB,OA)}\\ \text{res-val-eq(OA,OC)}\end{array}}{\text{type}} \quad \text{(EQ-IF-EQ-THEN)}$$

$$\text{eq-if-eq-then(natident,ifr/zz,ifr/zz,res-val-eq/id)}$$

$$\frac{\text{eq-if-eq-then(natident,A,B,C)}}{\text{eq-if-eq-then(natident,ifr/ss(A),ifr/ss(B),C)}}$$

$$\frac{\begin{array}{c}\{\emptyset_v\} \vdash \text{G } \mathbf{wf}_v\\ \text{G1} \vdash \top <:_{\approx} \text{T} \dashv \text{G}\end{array}}{\text{type}} \quad \text{(INVERT-WF-EMPTY)}$$

$$\text{invert-wf-empty(wfv/empty,sub-tp/top(wf-tp/top))}$$

$$\frac{\begin{array}{c}\text{sub-tp-trans(ST1,ST2,ST)}\\ \text{invert-wf-empty(WV,ST1)}\end{array}}{\text{invert-wf-empty(wfv/sub(ST2,WV),ST)}}$$

$$\frac{\begin{array}{c}\text{TC::G} \vdash \text{TC} <:_{\approx} \text{T} \dashv \text{G}\\ \text{TC::G} \vdash \{\mathbf{def}\ \ \text{LNF} : \text{T3} \to \text{T4}\} \wedge \{\mathbf{val}\ \ \text{LNV} : \text{T1}\} \wedge \text{MT} <:_{\approx} \text{TC} \dashv \text{TC::G}\\ \text{TC::G} \vdash \text{TC} <:_{\approx} \text{TC} \dashv \text{TC::G}\\ \text{type-mem(M',MT)}\\ \text{wf-res(V1,MT::G,T1)}\\ \text{MT::G} \vdash \text{T1} <:_{\approx} \text{T1} \dashv \text{TC::G}\\ \text{T3::TC::G} \vdash \text{R} : \text{T4}\\ \text{wf-env(H,G)}\\ \text{res-clos(V1,LNF,R,LNV,H,VC)}\\ \text{wf-res(VC,G,T)}\end{array}}{\text{type}} \quad \text{(CLOS-SAFE-RES)}$$

$$\text{clos-safe-res(CST,IN,STFF,M,TV,STV,TS,WE,res-clos/timeout,CST0)wf-res/timeout}$$

$$\text{clos-safe-res(CST,IN,STFF,M,wf-res/some(V2),STV,TS,WE,res-clos/some,CST0)wf-res/some(wfv/sub(CST,wfv/f(IN,STFF,M}$$

$$\frac{\begin{array}{c}\text{wf-res(OV1,G,\{}\mathbf{val}\ \ \text{LNV : T\})}\\ \text{eval-sel-res(LNV,OV1,OV2)}\\ \text{wf-res(OV2,G,T)}\end{array}}{\text{type}} \quad \text{(EVAL-SEL-SAFE-RES)}$$

$$\frac{\begin{array}{c}\text{eq-res-val-some-wfv(WEQ,WV2,OWV2)}\\ \text{eq-if-eq-then(ID,IF,IF',WEQ)}\\ \text{eq-to-id(EQ',ID)}\\ \text{eq-sym(EQ,EQ')}\\ \text{wfv-widen(ST1,WV1,WV2)}\\ \text{sub-tp-inv-recv(STR,ST1)}\\ \text{invert-wf-rec(WVX,WV1,STR,EQ)}\end{array}}{\text{eval-sel-safe-res(wf-res/some(WVX),es/ok(IF),OWV2)}}$$

$$\frac{\begin{array}{c}\text{no-wf-res(CONTRA,res-val/stuck,G,T,OWV)}\\ \text{no-subtype2(ST,incompat/top-recv,CONTRA)}\\ \text{invert-wf-empty(WVX,ST)}\end{array}}{\text{eval-sel-safe-res(wf-res/some(WVX),es/emptyfail,OWV)}}$$

$$\text{eval-sel-safe-res(wf-res/timeout,es/timeout,wf-res/timeout)}$$

$$\frac{\begin{array}{c}\text{wf-res(OVF,G0,\{}\textbf{def}\ \text{ LNF : T1} \rightarrow \text{T2\})}\\ \text{wf-res(OVX,G0,T1)}\\ \text{eval-app-res(N,LNF,OVF,OVX,OVA)}\\ \text{wf-res(OVA,G0,T2)}\end{array}}{\text{type}} \quad \text{(EVAL-APP-SAFE-RES)}$$

$$\frac{\begin{array}{c}\text{G} \vdash \text{E : T}\\ \text{wf-env(H,G)}\\ \text{eval-exp-res(N,H,E,OV)}\\ \text{wf-res(OV,G,T)}\end{array}}{\text{type}} \quad \text{(EVAL-SAFE-RES)}$$

$$\frac{\begin{array}{c}\text{eq-wf-res(WEQ,Z',Z")}\\ \text{eq-if-eq-then(ID,IF,IF',WEQ)}\\ \text{eq-to-id(EQ',ID)}\\ \text{eq-sym(EQ,EQ')}\\ \text{wf-res-widen(STR',Z,Z')}\\ \text{extend-sub-tp1(STR,X7,STR')}\\ \text{eval-safe-res(TR,wfe/c(wfe/c(WE,WXX'),WVX'),EVA,Z)}\\ \text{eq-wfv-clos(EQ,WXX,WXX')}\\ \text{wfv-widen(STA',WVX,WVX')}\\ \text{extend-sub-tp2(STA,X7,STA')}\\ \text{sub-tp-inv-fun(ST,STA,STR)}\\ \text{invert-wf-fun(WVF,TR,WE,ST,WXX,EQ)}\end{array}}{\text{eval-app-safe-res(wf-res/some(WVF),wf-res/some(WVX),ea/ok(IF,EVA),Z")}}$$

$$\frac{\begin{array}{c}\text{no-wf-res(CONTRA,res-val/stuck,G,T,OWA)}\\ \text{no-subtype2(ST,incompat/top-arrow,CONTRA)}\\ \text{invert-wf-empty(WVF,ST)}\end{array}}{\text{eval-app-safe-res(wf-res/some(WVF),X6,ea/emptyfail,OWA)}}$$

$$\text{eval-app-safe-res(X7,X8,ea/timeout,wf-res/timeout)}$$

$$\text{eval-app-safe-res(X7,X8,ea/timeout1,wf-res/timeout)}$$

$$\text{eval-app-safe-res(X7,X8,ea/timeout2,wf-res/timeout)}$$

$$\text{eval-safe-res(X5,X6,er/timeout,wf-res/timeout)}$$

$$\frac{\begin{array}{c}\text{to-case-some(res-val-get/some,VW,V)}\\ \text{eval-empty-safe(ZT,VW)}\end{array}}{\text{eval-safe-res(ZT,X5,er/empty,V)}}$$

$$\frac{\begin{array}{c}\text{wf-res-widen(ST,V,V2)}\\ \text{to-case-some(VG,VW,V)}\\ \text{lookup-safe-res(E,L,L2,VG,VW)}\\ \text{invert-var(ZT,L,ST)}\end{array}}{\text{eval-safe-res(ZT,E,er/var(L2),V2)}}$$

$$\frac{\begin{array}{c}\text{wf-res-widen(ST,WVS,WVS2)}\\ \text{eval-sel-safe-res(WVX,EVS,WVS)}\\ \text{eval-safe-res(TS,WE,EVR,WVX)}\\ \text{invert-sel(ZT,TS,ST)}\end{array}}{\text{eval-safe-res(ZT,WE,er/sel(EVS,EVR),WVS2)}}$$

$$\frac{\begin{array}{c}\text{clos-safe-res(CST,IN,STFF,M,TV,STV,TS,WE,RX,CST0)}WV\\ \text{eval-safe-res(TSV,wfe/c(WE,wfv/f(sub-tp/and1b(wf-tp/fun(wf-tp/top,wf-tp/bot),sub-tp/and1b(wf-tp/recv(wf-tp/top),STM)),}\\ \text{sub-tp-refl(WR,STM)}\\ \text{sub-tp-refl(WFF,STFF)}\\ \text{extract-wf(CST,WFF,X18)}\\ \text{invert-fun(ZT,TS,TSV,M,WR,STV,CST,IN)}\end{array}}{\text{eval-safe-res(ZT,WE,er/fun(RX,E2),WVF)}}$$

$$\frac{\begin{array}{c}\text{wf-res-widen(ST,WVA,WVA')}\\ \text{eval-app-safe-res(ZF,ZX,EVA,WVA)}\\ \text{eval-safe-res(TF,WE,EVF,ZF)}\\ \text{eval-safe-res(TX,WE,EVX,ZX)}\\ \text{invert-app(ZT,TF,TX,ST)}\end{array}}{\text{eval-safe-res(ZT,WE,er/app(EVA,EVX,EVF),WVA')}}$$