

# Type Reconstruction and Polymorphism

Week 9

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# Type Checking and Type Reconstruction

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We now come to the question of type checking and type reconstruction.

**Type checking:** Given  $\Gamma$ ,  $t$  and  $T$ , check whether  $\Gamma \vdash t : T$

**Type reconstruction:** Given  $\Gamma$  and  $t$ , find a type  $T$  such that  $\Gamma \vdash t : T$

Type checking and reconstruction seem difficult since parameters in lambda calculus do not carry their types with them.

Type reconstruction also suffers from the problem that a term can have many types.

**Idea:** : We construct all type derivations in parallel, reducing type reconstruction to a unification problem.

# From Judgements to Equations

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$TP : \text{Judgement} \rightarrow \text{Equations}$

$TP(\Gamma \vdash t : T) =$

**case  $t$  of**

$x$  :  $\{\Gamma(x) \hat{=} T\}$

$\lambda x.t'$  : **let  $a, b$  fresh in**

$\{(a \rightarrow b) \hat{=} T\} \cup$

$TP(\Gamma, x : a \vdash t' : b)$

$t t'$  : **let  $a$  fresh in**

$TP(\Gamma \vdash t : a \rightarrow T) \cup$

$TP(\Gamma \vdash t' : a)$

# Constants

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Constants are treated as variables in the initial environment.

However, we have to make sure we create a new instance of their type as follows:

$$\begin{aligned} \mathit{newInstance}(\forall a_1, \dots, a_n. S) = & \\ & \mathbf{let } b_1, \dots, b_n \mathbf{ fresh in} \\ & [b_1/a_1, \dots, b_n/a_n]S \\ \mathit{TP}(\Gamma \vdash t : T) = & \\ & \mathbf{case } t \mathbf{ of} \\ & x \quad : \quad \{ \mathit{newInstance}(\Gamma(x)) \hat{=} T \} \\ & \dots \end{aligned}$$

# Soundness and Completeness I

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**Definition:** In general, a type reconstruction algorithm  $\mathcal{A}$  assigns to an environment  $\Gamma$  and a term  $t$  a set of types  $\mathcal{A}(\Gamma, t)$ .

The algorithm is **sound** if for every type  $T \in \mathcal{A}(\Gamma, t)$  we can prove the judgement  $\Gamma \vdash t : T$ .

The algorithm is **complete** if for every provable judgement  $\Gamma \vdash t : T$  we have that  $T \in \mathcal{A}(\Gamma, t)$ .

**Theorem:**  $TP$  is sound and complete. Specifically:

$$\Gamma \vdash t : T \quad \text{iff} \quad \exists \bar{b}. [T/a]EQNS$$

*where*

$a$  is a new type variable

$$EQNS = TP(\Gamma \vdash t : a)$$

$$\bar{b} = tv(EQNS) \setminus tv(\Gamma)$$

Here,  $tv$  denotes the set of free type variables (of a term, and environment, an equation set).

# Type Reconstruction and Unification

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**Problem:** : Transform set of equations

$$\{T_i \hat{=} U_i\}_{i=1, \dots, m}$$

into equivalent substitution

$$\{a_j \hat{=} T'_j\}_{j=1, \dots, n}$$

where type variables do not appear recursively on their right hand sides (directly or indirectly). That is:

$$a_j \notin tv(T'_k) \quad \text{for } j = 1, \dots, n, k = j, \dots, n$$

# Substitutions

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A **substitution**  $s$  is an idempotent mapping from type variables to types which maps all but a finite number of type variables to themselves.

We often represent a substitution as a set of equations  $a \hat{=} T$  with  $a$  not in  $tv(T)$ .

Substitutions can be generalized to mappings from types to types by defining

$$\begin{aligned} s(T \rightarrow U) &= sT \rightarrow sU \\ s(K[T_1, \dots, T_n]) &= K[sT_1, \dots, sT_n] \end{aligned}$$

Substitutions are idempotent mappings from types to types, i.e.  $s(s(T)) = s(T)$ . (why?)

The  $\circ$  operator denotes composition of substitutions (or other functions):  $(f \circ g) x = f(gx)$ .



# A Unification Algorithm

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We present an incremental version of Robinson's algorithm (1965).

$$\begin{aligned}mgu & : (Type \hat{=} Type) \rightarrow Subst \rightarrow Subst \\mgu(T \hat{=} U) s & = mgu'(sT \hat{=} sU) s \\mgu'(a \hat{=} a) s & = s \\mgu'(a \hat{=} T) s & = s \cup \{a \hat{=} T\} \quad \text{if } a \notin tv(T) \\mgu'(T \hat{=} a) s & = s \cup \{a \hat{=} T\} \quad \text{if } a \notin tv(T) \\mgu'(T \rightarrow T' \hat{=} U \rightarrow U') s & = (mgu(T' \hat{=} U') \circ mgu(T \hat{=} U)) s \\mgu'(K[T_1, \dots, T_n] \hat{=} K[U_1, \dots, U_n]) s & \\ & = (mgu(T_n \hat{=} U_n) \circ \dots \circ mgu(T_1 \hat{=} U_1)) s \\mgu'(T \hat{=} U) s & = error \quad \text{in all other cases}\end{aligned}$$

# Soundness and Completeness of Unification

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**Definition:** A substitution  $u$  is a **unifier** of a set of equations  $\{T_i \hat{=} U_i\}_{i=1, \dots, m}$  if  $uT_i = uU_i$ , for all  $i$ . It is a **most general unifier** if for every other unifier  $u'$  of the same equations there exists a substitution  $s$  such that  $u' = s \circ u$ .

**Theorem:** Given a set of equations  $EQNS$ . If  $EQNS$  has a unifier then  $mgu\ EQNS\ \{\}$  computes the most general unifier of  $EQNS$ . If  $EQNS$  has no unifier then  $mgu\ EQNS\ \{\}$  fails.

# From Judgements to Substitutions

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$TP : \text{Judgement} \rightarrow \text{Subst} \rightarrow \text{Subst}$

$TP(\Gamma \vdash t : T) =$

**case  $t$  of**

$x$  : **mgu**( $\text{newInstance}(\Gamma x) \hat{=} T$ )

$\lambda x.t'$  : **let**  $t, u$  **fresh in**

**mgu**(( $t \rightarrow u$ )  $\hat{=} T$ )  $\circ$

$TP(\Gamma, x : t \vdash t' : u)$

$t t'$  : **let**  $t$  **fresh in**

$TP(\Gamma \vdash t : a \rightarrow T)$   $\circ$

$TP(\Gamma \vdash t' : a)$

# Soundness and Completeness II

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One can show by comparison with the previous algorithm:

**Theorem:**  $TP$  is sound and complete. Specifically:

$$\Gamma \vdash t : T \quad \text{iff} \quad T = r(s(t))$$

**where**

$t$  is a new type variable

$$s = TP (\Gamma \vdash t : t) \{ \}$$

$r$  is a substitution on  $tv(s t) \setminus tv(s \Gamma)$

# Strong Normalization

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**Question:** Can  $\Omega$  be given a type?

$$\Omega = (\lambda x.xx)(\lambda x.xx) :?$$

What about  $Y$ ?

Self-application is not typable!

In fact, we have more:

**Theorem:** (Strong Normalization) If  $\vdash t : T$ , then there is a value  $V$  such that  $t \rightarrow^* V$ .

**Corollary:** Simply typed lambda calculus is not Turing complete.

# Polymorphism

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In the simply typed lambda calculus, a term can have many types. But a variable or parameter has only one type.

Example:

$$(\lambda x.xx)(\lambda y.y)$$

is untypable. But if we substitute actual parameter for formal, we obtain

$$(\lambda y.y)(\lambda y.y) : a \rightarrow a$$

Functions which can be applied to arguments of many types are called **polymorphic**.

# Polymorphism in Programming

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Polymorphism is essential for many program patterns.

Example: map

```
def map f xs =  
  if (isEmpty (xs)) nil  
  else cons (f (head xs)) (map (f, tail xs))
```

...

```
names: List[String]
```

```
nums : List[Int]
```

...

```
map toUpperCase names
```

```
map increment nums
```

Without a polymorphic type for map one of the last two lines is always illegal!

# Forms of Polymorphism

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Polymorphism means “having many forms”.

Polymorphism also comes in several forms.

- **Universal polymorphism**, sometimes also called **generic types**: The ability to instantiate type variables.
- **Inclusion polymorphism**, sometimes also called **subtyping**: The ability to treat a value of a subtype as a value of one of its supertypes.
- **Ad-hoc polymorphism**, sometimes also called **overloading**: The ability to define several versions of the same function name, with different types.

We first concentrate on universal polymorphism.

Two basic approaches: **explicit** or **implicit**.



# Explicit Polymorphism

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We introduce a polymorphic type  $\forall a.T$ , which can be used just as any other type.

We then need to make introduction and elimination of  $\forall$ 's explicit.

Typing rules:

$$\begin{array}{c} (\forall E) \frac{\Gamma \vdash t : \forall a.T}{\Gamma \vdash t[U] : [U/a]T} \qquad (\forall I) \frac{\Gamma \vdash t : T}{\Gamma \vdash \Lambda a.t : \forall a.T} \end{array}$$

We also need to give all parameter types, so programs become verbose.

**Example:**

```
def map [a][b] (f: a -> b) (xs: List[a]) =
  if (isEmpty [a] (xs)) nil [a]
  else cons [b] (f (head [a] xs)) (map [a][b] (f, tail [a] xs))
...
names: List[String]
nums : List[Int]
...
map [String] [String] toUpperCase names
map [Int] [Int] increment nums
```

# Implicit Polymorphism

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Implicit polymorphism does not require annotations for parameter types or type instantiations.

**Idea:** In addition to types (as in simply typed lambda calculus), we have a new syntactic category of **type schemes**. Syntax:

$$\text{Type Scheme } S ::= T \mid \forall a.S$$

Type schemes are not fully general types; they are used only to type named values, introduced by a `val` construct.

The resulting type system is called the **Hindley/Milner system**, after its inventors. (The original treatment uses `let ... in ...` rather than `val ... ; ...`).

# Hindley/Milner Typing rules

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$$(\text{VAR}) \quad \Gamma, x : S, \Gamma' \vdash x : S \quad (x \notin \text{dom}(\Gamma'))$$

$$(\forall\text{E}) \frac{\Gamma \vdash t : \forall a.T}{\Gamma \vdash t : [U/a]T} \quad (\forall\text{I}) \frac{\Gamma \vdash t : T \quad a \notin \text{tv}(\Gamma)}{\Gamma \vdash t : \forall a.T}$$

$$(\text{LET}) \frac{\Gamma \vdash t : S \quad \Gamma, x : S \vdash t' : T}{\Gamma \vdash \mathbf{let} \ x = t \ \mathbf{in} \ t' : T}$$

The other two rules are as in simply typed lambda calculus:

$$(\rightarrow\text{I}) \frac{\Gamma, x : T \vdash t : U}{\Gamma \vdash \lambda x.t : T \rightarrow U} \quad (\rightarrow\text{E}) \frac{\Gamma \vdash M : T \rightarrow U \quad \Gamma \vdash N : T}{\Gamma \vdash M N : U}$$

# Hindley/Milner in Programming Languages

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Here is a formulation of the map example in the Hindley/Milner system.

```
let map =  $\lambda f.\lambda xs$  in
  if (isEmpty (xs)) nil
  else cons (f (head xs)) (map (f, tail xs))
...
// names: List[String]
// nums : List[Int]
// map  :  $\forall a.\forall b.(a \rightarrow b) \rightarrow List[a] \rightarrow List[b]$ 
...
map toUpperCase names
map increment nums
```

## Limitations of Hindley/Milner

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Hindley/Milner still does not allow parameter types to be polymorphic.  
I.e.

$$(\lambda x.xx)(\lambda y.y)$$

is still ill-typed, even though the following is well-typed:

$$\mathbf{let\ id = \lambda y.y\ in\ id\ id}$$

With explicit polymorphism the expression could be completed to a well-typed term:

$$(\Lambda a.\lambda x : (\forall a : a \rightarrow a).x[a \rightarrow a](x[a]))(\Lambda b.\lambda y.y)$$

# The Essence of let

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We regard

*let*  $x = t$  *in*  $t'$

as a shorthand for

$[t/x]t'$

We use this equivalence to get a revised Hindley/Milner system.

**Definition:** Let  $HM'$  be the type system that results if we replace rule (LET) from the Hindley/Milner system  $HM$  by:

$$\text{(LET')} \frac{\Gamma \vdash t : T \quad \Gamma \vdash [t/x]t' : U}{\Gamma \vdash \text{let } x = t \text{ in } t' : U}$$

**Theorem:**  $\Gamma \vdash_{HM} t : S$  iff  $\Gamma \vdash_{HM'} t : S$

The theorem establishes the following connection between the Hindley/Milner system and the simply typed lambda calculus  $F_1$ :

**Corollary:** Let  $t^*$  be the result of expanding all *let*'s in  $t$  according to the rule

$$\mathbf{let} \ x = t \ \mathbf{in} \ t' \ \rightarrow \ [t/x]t'$$

Then

$$\Gamma \vdash_{HM} t : T \ \Rightarrow \ \Gamma \vdash_{F_1} t^* : T$$

Furthermore, if every *let*-bound name is used at least once, we also have the reverse:

$$\Gamma \vdash_{F_1} t^* : T \ \Rightarrow \ \Gamma \vdash_{HM} t : T$$



# Principal Types

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**Definition:** A type  $T$  is a **generic instance** of a type scheme  $S = \forall\alpha_1 \dots \forall\alpha_n.T'$  if there is a substitution  $s$  on  $\alpha_1, \dots, \alpha_n$  such that  $T = sT'$ . We write in this case  $S \leq T$ .

**Definition:** A type scheme  $S'$  is a generic instance of a type scheme  $S$  iff for all types  $T$

$$S' \leq T \Rightarrow S \leq T$$

We write in this case  $S \leq S'$ .

**Definition:** A type scheme  $S$  is **principal** (or: **most general**) for  $\Gamma$  and  $t$  iff

- $\Gamma \vdash t : S$
- $\Gamma \vdash t : S'$  implies  $S \leq S'$

**Definition:** A type system  $TS$  has the **principal typing property** iff, whenever  $\Gamma \vdash_{TS} t : S$  then there exists a principal type scheme for  $\Gamma$  and  $t$ .

**Theorem:**

1.  $HM'$  without **let** has the p.t.p.
2.  $HM'$  with **let** has the p.t.p.
3.  $HM$  has the p.t.p.

Proof sketch: (1.): Use type reconstruction result for the simply typed lambda calculus. (2.): Expand all **let**'s and apply (1.). (3.): Use equivalence between  $HM$  and  $HM'$ .

These observations could be used to come up with a type reconstruction algorithm for  $HM$ . But in practice one takes a more direct approach.

# Type Reconstruction for Hindley/Milner

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Type reconstruction for the Hindley/Milner system works as for simply typed lambda calculus. We only have to add a clause for *let* expressions:

$TP : \text{Judgement} \rightarrow \text{Subst} \rightarrow \text{Subst}$

$TP(\Gamma \vdash t : T) s =$

**case  $t$  of**

...

**let  $x = t_1$  in  $t_2$  : let  $a, b$  fresh in**

**let  $s_1 = TP(\Gamma \vdash t_1 : a)$  in**

**$TP(\Gamma, x : \mathbf{gen}(s_1 \Gamma, s_1 a) \vdash t_2 : b) s_1$**

where  $\mathbf{gen}(\Gamma, T) = \forall tv(T) \setminus tv(\Gamma). T$ .

