Foundations of Software Winter Semester 2007

Week 11

November 27

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Motivation

With our usual typing rule for applications

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$

the term

(λ r:{x:Nat}. r.x) {x=0,y=1}

is not well typed.

Motivation

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$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$

the term

(λ r:{x:Nat}. r.x) {x=0,y=1}

Subtyping

is not well typed.

But this is silly: all we're doing is passing the function a *better* argument than it needs.

Similarly, in object-oriented languages, we want to be able to define hierarchies of classes, with classes lower in the hierarchy having richer interfaces than their ancestors higher in the hierarchy, and use instances of richer classes in situations where one of their ancestors are expected.

Subsumption

We achieve the effect we want by:

- 1. a subtyping relation between types, written S <: T
- 2. a rule of *subsumption* stating that, if S <: T, then any value of type S can also be regarded as having type T

$$\frac{\Gamma \vdash t : S \quad S <: T}{\Gamma \vdash t : T}$$
(T-SUB)

Example

We will define subtyping between record types so that, for example,

{x:Nat, y:Nat} <: {x:Nat}</pre>

So, by subsumption,

⊢ {x=0,y=1} : {x:Nat}

and hence

(λ r:{x:Nat}. r.x) {x=0,y=1}

is well typed.

The Subtype Relation: Records

"Width subtyping" (forgetting fields on the right):

 $\{l_i:T_i \stackrel{i\in 1..n+k}{<:} \{l_i:T_i \stackrel{i\in 1..n}{>} \}$ (S-RCDWIDTH)

Intuition: $\{x: Nat\}$ is the type of all records with *at least* a numeric x field.

Note that the record type with *more* fields is a *sub*type of the record type with fewer fields.

Reason: the type with more fields places a *stronger constraint* on values, so it describes *fewer values*.

The Subtype Relation: Records

Permutation of fields:

$$\frac{\{k_j: S_j \ ^{j \in 1..n}\} \text{ is a permutation of } \{1_i: T_i \ ^{i \in 1..n}\}}{\{k_j: S_j \ ^{j \in 1..n}\} <: \{1_i: T_i \ ^{i \in 1..n}\}} \text{ (S-RCDPERM)}$$

By using S-RCDPERM together with S-RCDWIDTH and S-TRANS allows us to drop arbitrary fields within records.

The Subtype Relation: Records

"Depth subtyping" within fields:

 $\frac{\text{for each } i \quad S_i \leq T_i}{\{\mathbf{l}_i: S_i \stackrel{i \in 1..n}{\leq} \leq \{\mathbf{l}_i: T_i \stackrel{i \in 1..n}{\leq}\}} \quad (S-\text{RCDDEPTH})$

The types of individual fields may change.

{a:Nat,b:Nat} <: {a:	S-RCDWIDTH	{m:Nat}	
<pre>{x:{a:Nat,b:Nat},y</pre>	:{m:Nat}} <: {	x:{a:Nat},y:	S-RCDDEPTH

Variations

Real languages often choose not to adopt all of these record subtyping rules. For example, in Java,

- A subclass may not change the argument or result types of a method of its superclass (i.e., no depth subtyping)
- Each class has just one superclass ("single inheritance" of classes)

 \longrightarrow each class member (field or method) can be assigned a single index, adding new indices "on the right" as more members are added in subclasses (i.e., no permutation for classes)

 A class may implement multiple *interfaces* ("multiple inheritance" of interfaces)
 I.e., permutation is allowed for interfaces.

The Subtype Relation: Arrow types

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \rightarrow S_2 <: T_1 \rightarrow T_2}$$
 (S-Arrow)

Note the order of T_1 and S_1 in the first premise. The subtype relation is *contravariant* in the left-hand sides of arrows and *covariant* in the right-hand sides.

Intuition: if we have a function f of type $S_1 \rightarrow S_2$, then we know that f accepts elements of type S_1 ; clearly, f will also accept elements of any subtype T_1 of S_1 . The type of f also tells us that it returns elements of type S_2 ; we can also view these results belonging to any supertype T_2 of S_2 . That is, any function f of type $S_1 \rightarrow S_2$ can also be viewed as having type $T_1 \rightarrow T_2$.

		2 •	elation: General rules	
			S <: S	(S-Refl)
S <: Top	(S-Top)		$\frac{S <: 0 \qquad 0 <: T}{S <: T}$	(S-TRANS)
	of the subtype relation. S <: Top	S <: Top (S-TOP)	of the subtype relation. S <: Top (S-TOP)	of the subtype relation. $S \le Top$ (S-TOP) $\frac{S \le U U \le T}{S \le T}$

Subtype relation		
	S <: S	(S-REFL)
	S <: U U <: T S <: T	(S-Trans)
{1,	$_{i}:T_{i} \stackrel{i \in 1n+k}{<:} \{1_{i}:T_{i} \stackrel{i \in 1}{:}$	"} (S-RCDWIDTH)
{]	for each i $S_i \leq T_i$ $l_i:S_i \stackrel{i \in 1n}{:::::::::::::::::::::::::::::::::::$	(S-RCDDEPTH)
$\frac{\{\mathbf{k}_j: \mathbf{S}_j \in \mathbb{N}, \mathbf{s}_j\} \text{ is } \mathbf{s}_j}{\{\mathbf{k}_j: \mathbf{S}_j\}}$	a permutation of $\{l_i: T_i\}^{i \in 1n}$ <: $\{l_i: T_i\}^{i \in 1n}$	(S-RCDPERM)
	$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \rightarrow S_2 <: T_1 \rightarrow T_2}$	(S-Arrow)
	S <: Top	(S-Top)

Aside: Structural vs. declared subtyping

The subtype relation we have defined is *structural*: We decide whether S is a subtype of T by examining the structure of S and T.

By contrast, the subtype relation in most OO languages (e.g., Java) is *explicitly declared*: S is a subtype of T only if the programmer has stated that it should be.

There are pragmatic arguments for both.

For the moment, we'll concentrate on structural subtyping, which is the more fundamental of the two. (It is sound to *declare* S to be a subtype of T only when S is structurally a subtype of T.)

We'll come back to declared subtyping when we talk about Featherweight Java.

Properties of Subtyping

Safety

Statements of progress and preservation theorems are unchanged from λ_{\rightarrow} .

Proofs become a bit more involved, because the typing relation is no longer syntax directed.

Given a derivation, we don't always know what rule was used in the last step. The rule T-SUB could appear anywhere.

$$\frac{\Gamma \vdash t : S \quad S <: T}{\Gamma \vdash t : T}$$
(T-SUB)

An Inversion Lemma for Subtyping

Lemma: If U <: $T_1 \rightarrow T_2$, then U has the form $U_1 \rightarrow U_2$, with $T_1 <: U_1$ and $U_2 \leq T_2$.

Proof: By induction on subtyping derivations.

An Inversion Lemma for Subtyping

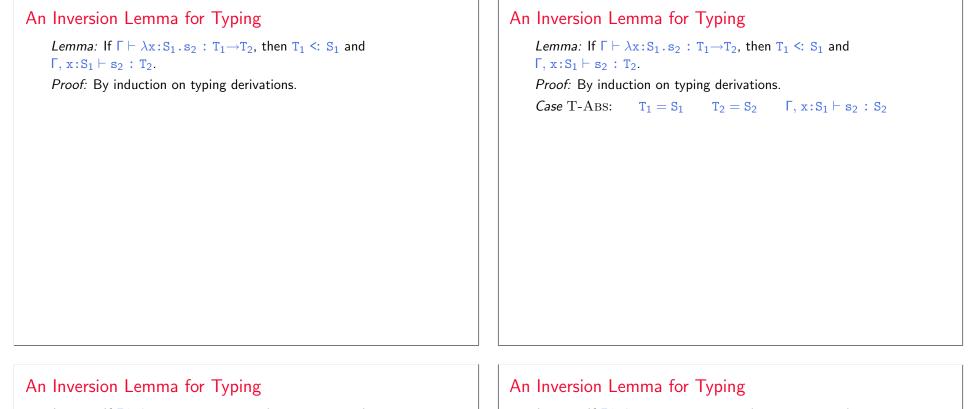
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Proof: By induction on subtyping derivations.

Case S-ARROW: $U = U_1 \rightarrow U_2$ $T_1 \lt: U_1$ $U_2 \lt: T_2$

An Inversion Lemma for Subtyping	An Inversion Lemma for Subtyping
<i>Lemma:</i> If U <: $T_1 \rightarrow T_2$, then U has the form $U_1 \rightarrow U_2$, with $T_1 <: U_1$ and $U_2 <: T_2$.	<i>Lemma:</i> If U <: $T_1 \rightarrow T_2$, then U has the form $U_1 \rightarrow U_2$, with $T_1 <: U_1$ and $U_2 <: T_2$.
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Immediate.	Immediate.
	Case S-Refl: $U = T_1 \rightarrow T_2$
An Inversion Lomma for Subturing	An Inversion Lomma for Subturing
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 $\begin{array}{l} \textit{Lemma: If } \Gamma \vdash \lambda x : S_1 . \, s_2 \, : \, T_1 {\rightarrow} T_2 \text{, then } T_1 <: \, S_1 \text{ and } \\ \Gamma, \, x : S_1 \vdash s_2 \, : \, T_2. \end{array}$

Proof: By induction on typing derivations.

 $\label{eq:case T-ABS: T_1 = S_1 T_2 = S_2 } \quad \ \ \Gamma, \ x \colon S_1 \vdash s_2 \, \colon \, S_2 \\ \mbox{Immediate.}$

Case T-SUB: $\Gamma \vdash \lambda x : S_1 . s_2 : U \qquad U \leq T_1 \rightarrow T_2$

n Inversion Lemma for Typing Lemma: If $\Gamma \vdash \lambda x : S_1 . s_2 : T_1 \rightarrow T_2$, then $T_1 \leq S_1$ and $\Gamma, x : S_1 \vdash s_2 : T_2$. Proof: By induction on typing derivations. Case T-ABS: $T_1 = S_1$ $T_2 = S_2$ $\Gamma, x : S_1 \vdash s_2 : S_2$ Immediate. Case T-SUB: $\Gamma \vdash \lambda x : S_1 . s_2 : U$ $U \leq T_1 \rightarrow T_2$ By the subtyping inversion lemma, $U = U_1 \rightarrow U_2$, with $T_1 \leq U_1$ and $U_2 \leq T_2$.

An Inversion Lemma for Typing Lemma: If $\Gamma \vdash \lambda x : S_1 . s_2 : T_1 \rightarrow T_2$, then $T_1 \le S_1$ and $\Gamma, x : S_1 \vdash s_2 : T_2$. Proof: By induction on typing derivations. Case T-ABS: $T_1 = S_1$ $T_2 = S_2$ $\Gamma, x : S_1 \vdash s_2 : S_2$ Immediate. Case T-SUB: $\Gamma \vdash \lambda x : S_1 . s_2 : U$ $U \le T_1 \rightarrow T_2$ By the subtyping inversion lemma, $U = U_1 \rightarrow U_2$, with $T_1 \le U_1$ and $U_2 \le T_2$. The IH now applies, yielding $U_1 \le S_1$ and $\Gamma, x : S_1 \vdash s_2 : U_2$.

An Inversion Lemma for Typing Lemma: If $\Gamma \vdash \lambda x : S_1 . s_2 : T_1 \rightarrow T_2$, then $T_1 \leq S_1$ and

 $\begin{array}{l} \label{eq:relation} \end{tabular} \en$

An Inversion Lemma for Typing

 $\begin{array}{l} \textit{Lemma:} \ \text{If } \Gamma \vdash \lambda x \colon S_1 \cdot s_2 \, \colon \, T_1 {\rightarrow} T_2, \ \text{then } T_1 \mathrel{\leq} S_1 \ \text{and} \\ \Gamma, x \colon S_1 \vdash s_2 \, \colon T_2. \end{array}$ $\begin{array}{l} \textit{Proof:} \ \text{By induction on typing derivations.} \\ \textit{Case } T\text{-}ABS: \quad T_1 = S_1 \quad T_2 = S_2 \quad \Gamma, x \colon S_1 \vdash s_2 \, \colon S_2 \\ \text{Immediate.} \\ \textit{Case } T\text{-}SUB: \quad \Gamma \vdash \lambda x \colon S_1 \cdot s_2 \, \colon U \quad U \mathrel{\leq} T_1 {\rightarrow} T_2 \\ \text{By the subtyping inversion lemma, } U = U_1 {\rightarrow} U_2, \ \text{with } T_1 \mathrel{\leq} U_1 \ \text{and} \\ U_2 \mathrel{\leq} T_2. \end{array}$ $\begin{array}{l} \text{The IH now applies, yielding } U_1 \mathrel{\leq} S_1 \ \text{and } \Gamma, x \colon S_1 \vdash s_2 \, \colon U_2. \\ \text{From } U_1 \mathrel{\leq} S_1 \ \text{and } T_1 \mathrel{\leq} U_2, \ \text{rule } S\text{-} TRANS \ \text{gives } T_1 \mathrel{\leq} S_1. \\ \text{From } \Gamma, x \colon S_1 \vdash s_2 \, \colon U_2 \ \text{and } U_2 \mathrel{\leq} T_2, \ \text{rule } T\text{-} SUB \ \text{gives} \\ \Gamma, x \colon S_1 \vdash s_2 \, \colon T_2, \ \text{and we are done.} \end{array}$

Preservation

Theorem: If $\Gamma \vdash t$: T and t \longrightarrow t', then $\Gamma \vdash t'$: T.

Proof: By induction on typing derivations.

eservation — subsumption case	Preservation — subsumption case
Case T-SUB: t : S S <: T	Case T-SUB: $t : S S \leq T$ By the induction hypothesis, $\Gamma \vdash t' : S$. By T-SUB, $\Gamma \vdash t : T$.

Preservation — application case

Case T-APP:

 $\mathtt{t} = \mathtt{t}_1 \ \mathtt{t}_2 \qquad \Gamma \vdash \mathtt{t}_1 \, : \, \mathtt{T}_{11} {\rightarrow} \mathtt{T}_{12} \qquad \Gamma \vdash \mathtt{t}_2 \, : \, \mathtt{T}_{11} \qquad \mathtt{T} = \mathtt{T}_{12}$

By the inversion lemma for evaluation, there are three rules by which t \longrightarrow t' can be derived: E-APP1, E-APP2, and E-APPABS. Proceed by cases.

Preservation — application case

Case T-APP:

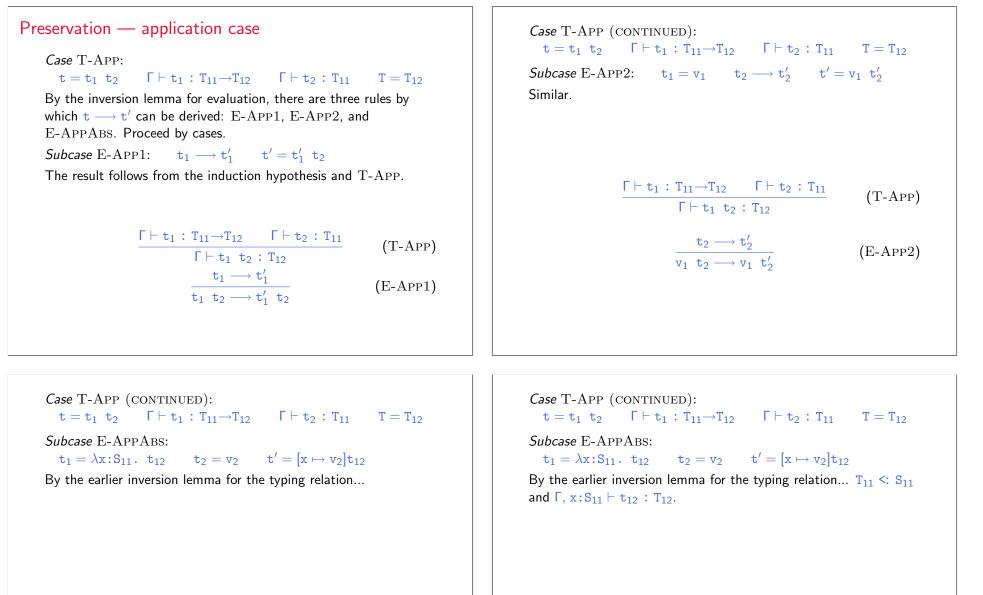
 $\mathtt{t} = \mathtt{t}_1 \ \mathtt{t}_2 \qquad \Gamma \vdash \mathtt{t}_1 \, : \, \mathtt{T}_{11} {\rightarrow} \mathtt{T}_{12} \qquad \Gamma \vdash \mathtt{t}_2 \, : \, \mathtt{T}_{11} \qquad \mathtt{T} = \mathtt{T}_{12}$

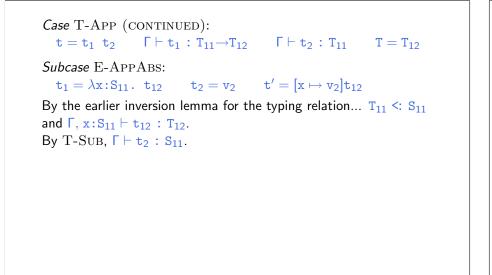
By the inversion lemma for evaluation, there are three rules by which t \longrightarrow t' can be derived: E-APP1, E-APP2, and E-APPABS. Proceed by cases.

Subcase E-APP1: $t_1 \longrightarrow t'_1$ $t' = t'_1 t_2$

The result follows from the induction hypothesis and T-APP.

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$



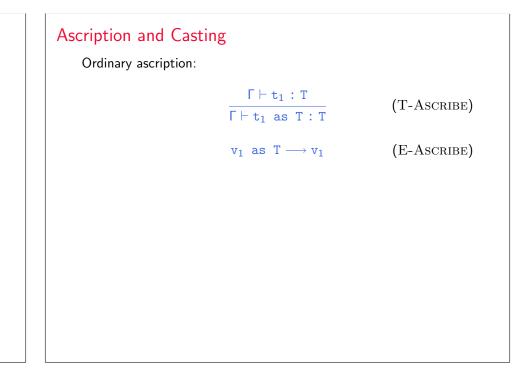


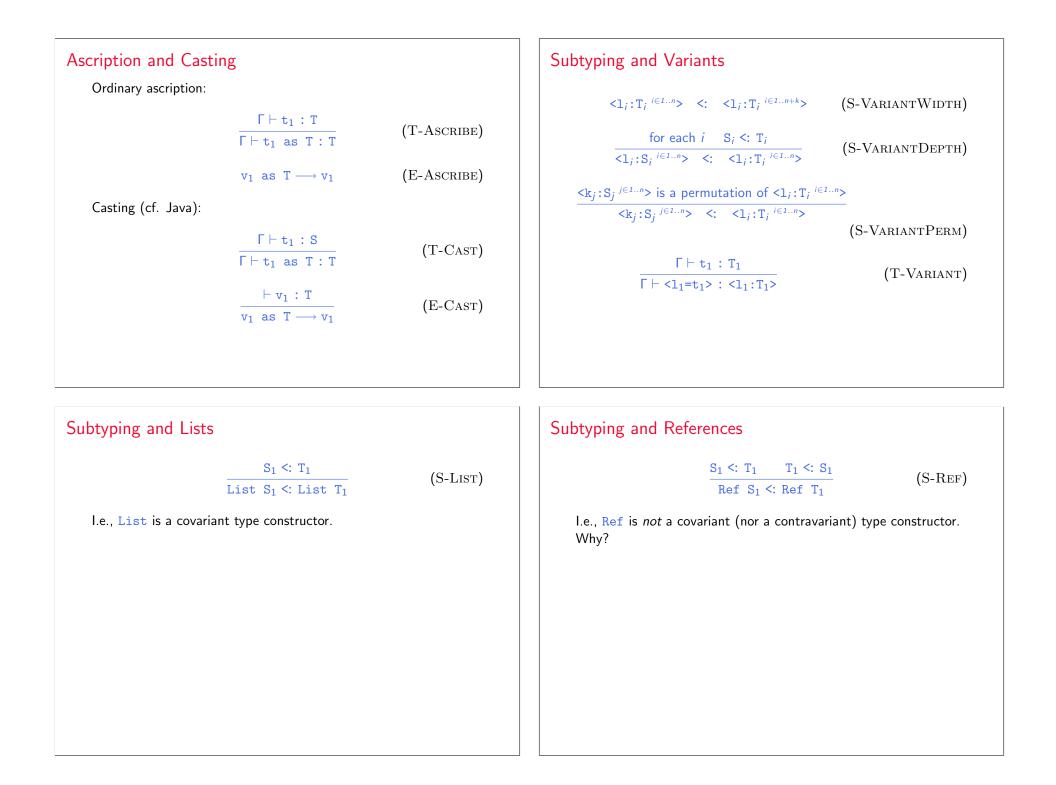
 $\begin{array}{ll} \mbox{Case T-APP (CONTINUED):} & t = t_1 \ t_2 & \Gamma \vdash t_1 : T_{11} {\rightarrow} T_{12} & \Gamma \vdash t_2 : T_{11} & T = T_{12} \\ \mbox{Subcase E-APPABS:} & & & \\ t_1 = \lambda x {:} S_{11} . \ t_{12} & t_2 = v_2 & t' = [x \mapsto v_2] t_{12} \\ \mbox{By the earlier inversion lemma for the typing relation...} & T_{11} {<:} S_{11} \\ \mbox{and } \Gamma, x {:} S_{11} \vdash t_{12} : T_{12}. \\ \mbox{By T-SUB, } \Gamma \vdash t_2 : S_{11}. \\ \mbox{By the substitution lemma, } \Gamma \vdash t' : T_{12}, \mbox{ and we are done.} \\ \end{array}$

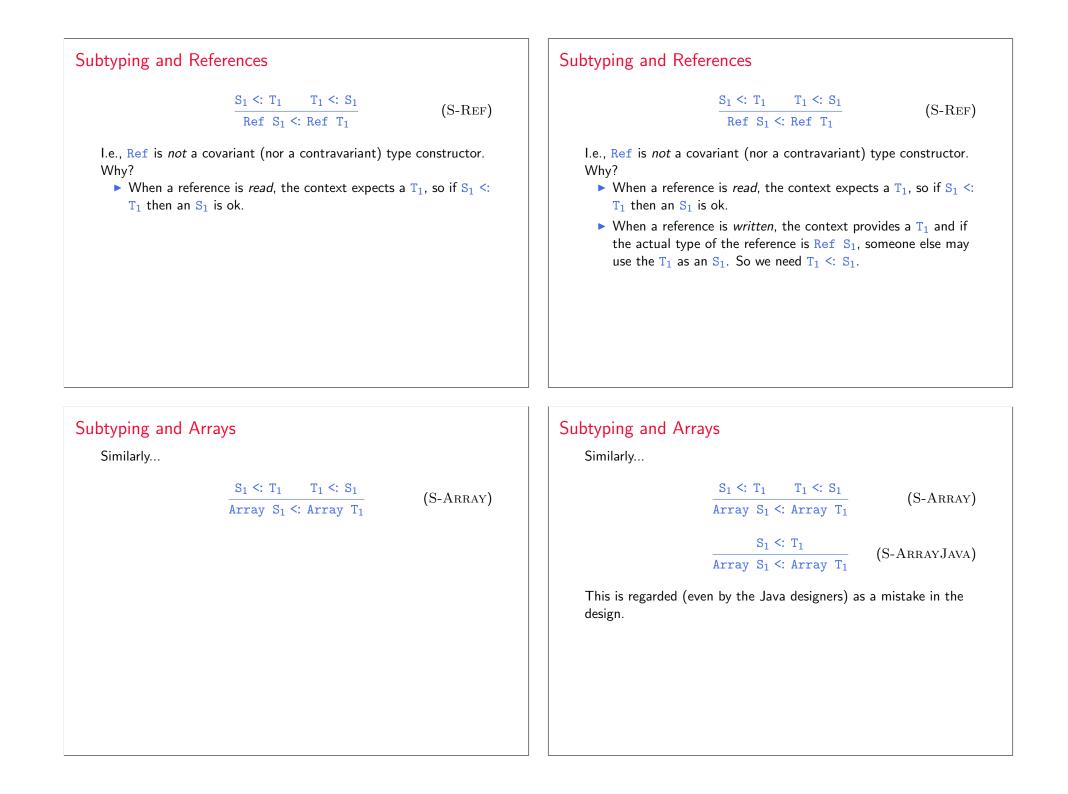
$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$

 $(\lambda x: T_{11}, t_{12}) \quad v_2 \longrightarrow [x \mapsto v_2] t_{12} \quad (E-APPABS)$

Subtyping with Other Features







References again Observation: a value of type Ref T can be used in two different ways: as a <i>source</i> for values of type T and as a <i>sink</i> for values of type T.	<pre>References again Observation: a value of type Ref T can be used in two different ways: as a source for values of type T and as a sink for values of type T. Idea: Split Ref T into three parts: Source T: reference cell with "read cabability" Sink T: reference cell with "write cabability" Ref T: cell with both capabilities</pre>
Modified Typing Rules $\frac{\Gamma \mid \Sigma \vdash t_1 : \text{Source } T_{11}}{\Gamma \mid \Sigma \vdash ! t_1 : T_{11}} \qquad (T-DEREF)$	Subtyping rules $\frac{S_1 <: T_1}{Source S_1 <: Source T_1}$ (S-SOURCE)
$\frac{\Gamma \mid \Sigma \vdash t_1 : \text{Sink } T_{11} \qquad \Gamma \mid \Sigma \vdash t_2 : T_{11}}{\Gamma \mid \Sigma \vdash t_1 := t_2 : \text{Unit}} \text{(T-Assign)}$	$\frac{T_1 <: S_1}{Sink S_1 <: Sink T_1} $ (S-SINK) Ref T ₁ <: Source T ₁ (S-REFSOURCE)
	Ref T ₁ <: Sink T ₁ (S-REFSINK)

Algorithmic Subtyping

Syntax-directed rules

In the simply typed lambda-calculus (without subtyping), each rule can be "read from bottom to top" in a straightforward way.

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$

If we are given some Γ and some t of the form t_1 t_2, we can try to find a type for t by

- 1. finding (recursively) a type for t_1
- 2. checking that it has the form $T_{11}{\rightarrow}T_{12}$

3. finding (recursively) a type for t_2

4. checking that it is the same as $T_{\rm 11}$

Technically, the reason this works is that We can divide the "positions" of the typing relation into *input positions* (Γ and t) and *output positions* (T).

- For the input positions, all metavariables appearing in the premises also appear in the conclusion (so we can calculate inputs to the "subgoals" from the subexpressions of inputs to the main goal)
- For the output positions, all metavariables appearing in the conclusions also appear in the premises (so we can calculate outputs from the main goal from the outputs of the subgoals)

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1 \ t_2 : T_{12}} \qquad (T-APP)$$

Syntax-directed sets of rules

The second important point about the simply typed lambda-calculus is that the *set* of typing rules is syntax-directed, in the sense that, for every "input" Γ and t, there one rule that can be used to derive typing statements involving t.

E.g., if t is an application, then we must proceed by trying to use T-APP. If we succeed, then we have found a type (indeed, the unique type) for t. If it fails, then we know that t is not typable.

 \longrightarrow no backtracking!

Non-syntax-directedness of typing

When we extend the system with subtyping, both aspects of syntax-directedness get broken.

1. The set of typing rules now includes *two* rules that can be used to give a type to terms of a given shape (the old one plus T-SUB)

$$\frac{\Gamma \vdash t : S \quad S <: T}{\Gamma \vdash t : T}$$
(T-SUB)

(Hence, if we translated the typing rules naively into a typechecking function, the case corresponding to T-SUB would cause divergence.)

What to do?

Non-syntax-directedness of subtyping

Moreover, the subtyping relation is not syntax directed either.

- 1. There are *lots* of ways to derive a given subtyping statement.
- 2. The transitivity rule

$$\frac{S <: U \quad U <: T}{S <: T}$$
 (S-Trans)

is badly non-syntax-directed: the premises contain a metavariable (in an "input position") that does not appear at all in the conclusion.

To implement this rule naively, we'd have to guess a value for ${\tt U}!$

What to do?

1. Observation: We don't *need* 1000 ways to prove a given typing or subtyping statement — one is enough.

 \longrightarrow Think more carefully about the typing and subtyping systems to see where we can get rid of excess flexibility

- 2. Use the resulting intuitions to formulate new "algorithmic" (i.e., syntax-directed) typing and subtyping relations
- 3. Prove that the algorithmic relations are "the same as" the original ones in an appropriate sense.

Conclusion

Polymorphism

Subtyping is a kind of *polymorphism*, which in Greek means "having many forms".

A *polymorphic* function may be applied to many different types of data.

Varieties of polymorphism:

- Parametric polymorphism (ML-style)
- Subtype polymorphism (OO-style)
- Ad-hoc polymorphism (overloading)