Type Systems
Winter Semester 2006

Week 8

December 6

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Plan

PREVIOUSLY: unit, sequencing, let

TODAY:

1. pairs, options, variants
2. recursion
3. state

NEXT: exceptions?
NEXT: polymorphic (not so simple) typing

Pairs

$\mathtt{t}$	::=	...	*terms*
		$\{\mathtt{t},\mathtt{t}\}$	*pair*
		$\mathtt{t}.1$	*first projection*
		$\mathtt{t}.2$	*second projection*
$\mathtt{v}$	::=	...	*values*
		$\{\mathtt{v},\mathtt{v}\}$	*pair value*
$\mathtt{T}$	::=	...	*types*
		$\mathtt{T}_1 \times \mathtt{T}_2$	*product type*

Evaluation rules for pairs

$$\{\mathtt{v}_1,\mathtt{v}_2\}.1 \longrightarrow \mathtt{v}_1 \qquad \text{(E-PairBeta1)}$$

$$\{\mathtt{v}_1,\mathtt{v}_2\}.2 \longrightarrow \mathtt{v}_2 \qquad \text{(E-PairBeta2)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1.1 \longrightarrow \mathtt{t}_1'.1} \qquad \text{(E-Proj1)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1.2 \longrightarrow \mathtt{t}_1'.2} \qquad \text{(E-Proj2)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\{\mathtt{t}_1,\mathtt{t}_2\} \longrightarrow \{\mathtt{t}_1',\mathtt{t}_2\}} \qquad \text{(E-Pair1)}$$

$$\frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2'}{\{\mathtt{v}_1,\mathtt{t}_2\} \longrightarrow \{\mathtt{v}_1,\mathtt{t}_2'\}} \qquad \text{(E-Pair2)}$$

Typing rules for pairs

$$\frac{\Gamma \vdash t_1 : T_1 \qquad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \{t_1, t_2\} : T_1 \times T_2} \quad \text{(T-Pair)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_1.1 : T_{11}} \quad \text{(T-Proj1)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_1.2 : T_{12}} \quad \text{(T-Proj2)}$$

Tuples

t	$::=$	...	*terms*
		$\{t_{i\ i \in 1..n}\}$	*tuple*
		$t.i$	*projection*
v	$::=$	...	*values*
		$\{v_{i\ i \in 1..n}\}$	*tuple value*
T	$::=$	...	*types*
		$\{T_{i\ i \in 1..n}\}$	*tuple type*

Evaluation rules for tuples

$$\{v_i \;^{i\in 1..n}\}.j \longrightarrow v_j \qquad \text{(E-ProjTuple)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.i \longrightarrow t_1'.i} \qquad \text{(E-Proj)}$$

$$\frac{t_j \longrightarrow t_j'}{\begin{array}{l}\{v_i \;^{i\in 1..j-1}, t_j, t_k \;^{k\in j+1..n}\} \\ \longrightarrow \{v_i \;^{i\in 1..j-1}, t_j', t_k \;^{k\in j+1..n}\}\end{array}} \qquad \text{(E-Tuple)}$$

Typing rules for tuples

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{t_i \;^{i\in 1..n}\} : \{T_i \;^{i\in 1..n}\}} \qquad \text{(T-Tuple)}$$

$$\frac{\Gamma \vdash t_1 : \{T_i \;^{i\in 1..n}\}}{\Gamma \vdash t_1.j : T_j} \qquad \text{(T-Proj)}$$

Records

$\mathtt{t}$	::=	...	*terms*
		$\{\mathtt{l}_i{=}\mathtt{t}_i\ ^{i\in 1..n}\}$	*record*
		$\mathtt{t}.\mathtt{l}$	*projection*
$\mathtt{v}$	::=	...	*values*
		$\{\mathtt{l}_i{=}\mathtt{v}_i\ ^{i\in 1..n}\}$	*record value*
$\mathtt{T}$	::=	...	*types*
		$\{\mathtt{l}_i{:}\mathtt{T}_i\ ^{i\in 1..n}\}$	*type of records*

Evaluation rules for records

$$\{\mathtt{l}_i{=}\mathtt{v}_i\ ^{i\in 1..n}\}.\mathtt{l}_j \longrightarrow \mathtt{v}_j \qquad \text{(E-ProjRcd)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1.\mathtt{l} \longrightarrow \mathtt{t}_1'.\mathtt{l}} \qquad \text{(E-Proj)}$$

$$\frac{\mathtt{t}_j \longrightarrow \mathtt{t}_j'}{\begin{array}{l}\{\mathtt{l}_i{=}\mathtt{v}_i\ ^{i\in 1..j-1}, \mathtt{l}_j{=}\mathtt{t}_j, \mathtt{l}_k{=}\mathtt{t}_k\ ^{k\in j+1..n}\} \\ \longrightarrow \{\mathtt{l}_i{=}\mathtt{v}_i\ ^{i\in 1..j-1}, \mathtt{l}_j{=}\mathtt{t}_j', \mathtt{l}_k{=}\mathtt{t}_k\ ^{k\in j+1..n}\}\end{array}} \qquad \text{(E-Rcd)}$$

Typing rules for records

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{l_i{=}t_i \ ^{i\in 1..n}\} : \{l_i{:}T_i \ ^{i\in 1..n}\}} \qquad \text{(T-Rcd)}$$

$$\frac{\Gamma \vdash t_1 : \{l_i{:}T_i \ ^{i\in 1..n}\}}{\Gamma \vdash t_1.l_j : T_j} \qquad \text{(T-Proj)}$$

Sums and variants

Sums – motivating example

```
PhysicalAddr = {firstlast:String, addr:String}
VirtualAddr  = {name:String, email:String}
Addr         = PhysicalAddr + VirtualAddr
inl  :  "PhysicalAddr → PhysicalAddr+VirtualAddr"
inr  :  "VirtualAddr → PhysicalAddr+VirtualAddr"
```

```
getName = λa:Addr.
  case a of
    inl x ⇒ x.firstlast
  | inr y ⇒ y.name;
```

New syntactic forms

t	::=	...	*terms*
		inl t	*tagging (left)*
		inr t	*tagging (right)*
		case t of inl x⇒t | inr x⇒t	*case*
v	::=	...	*values*
		inl v	*tagged value (left)*
		inr v	*tagged value (right)*
T	::=	...	*types*
		T+T	*sum type*

T_1+T_2 is a *disjoint union* of T_1 and T_2 (the tags inl and inr ensure disjointness)

New evaluation rules

$t \longrightarrow t'$

$$\texttt{case (inl } v_0\texttt{) of inl } x_1 \Rightarrow t_1 \texttt{ | inr } x_2 \Rightarrow t_2 \longrightarrow [x_1 \mapsto v_0]t_1 \quad \text{(E-CaseInl)}$$

$$\texttt{case (inr } v_0\texttt{) of inl } x_1 \Rightarrow t_1 \texttt{ | inr } x_2 \Rightarrow t_2 \longrightarrow [x_2 \mapsto v_0]t_2 \quad \text{(E-CaseInr)}$$

$$\frac{t_0 \longrightarrow t_0'}{\texttt{case } t_0 \texttt{ of inl } x_1 \Rightarrow t_1 \texttt{ | inr } x_2 \Rightarrow t_2 \longrightarrow \texttt{case } t_0' \texttt{ of inl } x_1 \Rightarrow t_1 \texttt{ | inr } x_2 \Rightarrow t_2} \quad \text{(E-Case)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{inl } t_1 \longrightarrow \texttt{inl } t_1'} \quad \text{(E-Inl)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{inr } t_1 \longrightarrow \texttt{inr } t_1'} \quad \text{(E-Inr)}$$

New typing rules

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \texttt{inl } t_1 : T_1 + T_2} \quad \text{(T-Inl)}$$

$$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \texttt{inr } t_1 : T_1 + T_2} \quad \text{(T-Inr)}$$

$$\frac{\Gamma \vdash t_0 : T_1 + T_2 \qquad \Gamma, x_1 : T_1 \vdash t_1 : T \qquad \Gamma, x_2 : T_2 \vdash t_2 : T}{\Gamma \vdash \texttt{case } t_0 \texttt{ of inl } x_1 \Rightarrow t_1 \texttt{ | inr } x_2 \Rightarrow t_2 : T} \quad \text{(T-Case)}$$

Sums and Uniqueness of Types

Problem:

> *If* `t` *has type* `T`*, then* `inl t` *has type* `T+U` *for* every `U`*.*

I.e., we've lost uniqueness of types.

Possible solutions:

- "Infer" `U` as needed during typechecking
- Give constructors different names and only allow each name to appear in one sum type (requires generalization to "variants," which we'll see next) — OCaml's solution
- Annotate each `inl` and `inr` with the intended sum type.

For simplicity, let's choose the third.

New syntactic forms

`t`	::=	...	*terms*
		`inl t as T`	*tagging (left)*
		`inr t as T`	*tagging (right)*
`v`	::=	...	*values*
		`inl v as T`	*tagged value (left)*
		`inr v as T`	*tagged value (right)*

Note that `as T` here is not the ascription operator that we saw before — i.e., not a separate syntactic form: in essence, there is an ascription "built into" every use of `inl` or `inr`.

New typing rules $\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \texttt{inl}\ t_1\ \texttt{as}\ T_1{+}T_2 : T_1{+}T_2} \quad \text{(T-Inl)}$$

$$\frac{\Gamma \vdash t_1 : T_2}{\Gamma \vdash \texttt{inr}\ t_1\ \texttt{as}\ T_1{+}T_2 : T_1{+}T_2} \quad \text{(T-Inr)}$$

Evaluation rules ignore annotations: $\boxed{t \longrightarrow t'}$

$$\begin{array}{l}\texttt{case (inl}\ v_0\ \texttt{as}\ T_0\texttt{)} \\ \texttt{of inl}\ x_1 \Rightarrow t_1\ \texttt{|}\ \texttt{inr}\ x_2 \Rightarrow t_2 \\ \qquad \longrightarrow [x_1 \mapsto v_0]t_1\end{array} \quad \text{(E-CaseInl)}$$

$$\begin{array}{l}\texttt{case (inr}\ v_0\ \texttt{as}\ T_0\texttt{)} \\ \texttt{of inl}\ x_1 \Rightarrow t_1\ \texttt{|}\ \texttt{inr}\ x_2 \Rightarrow t_2 \\ \qquad \longrightarrow [x_2 \mapsto v_0]t_2\end{array} \quad \text{(E-CaseInr)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{inl}\ t_1\ \texttt{as}\ T_2 \longrightarrow \texttt{inl}\ t_1'\ \texttt{as}\ T_2} \quad \text{(E-Inl)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{inr}\ t_1\ \texttt{as}\ T_2 \longrightarrow \texttt{inr}\ t_1'\ \texttt{as}\ T_2} \quad \text{(E-Inr)}$$

Variants

Just as we generalized binary products to labeled records, we can generalize binary sums to labeled *variants*.

New syntactic forms

`t`	::=	...	*terms*
		`<l=t> as T`	*tagging*
		`case t of` $\texttt{<l}_i\texttt{=x}_i\texttt{>}\Rightarrow\texttt{t}_i\ ^{i\in 1..n}$	*case*
`T`	::=	...	*types*
		$\texttt{<l}_i\texttt{:T}_i\ ^{i\in 1..n}\texttt{>}$	*type of variants*

New evaluation rules

$t \longrightarrow t'$

$$\texttt{case (<}l_j\texttt{=}v_j\texttt{> as T) of <}l_i\texttt{=}x_i\texttt{>}\Rightarrow t_i\ ^{i\in 1..n} \longrightarrow [x_j \mapsto v_j]t_j \qquad \text{(E-CaseVariant)}$$

$$\frac{t_0 \longrightarrow t_0'}{\texttt{case } t_0 \texttt{ of <}l_i\texttt{=}x_i\texttt{>}\Rightarrow t_i\ ^{i\in 1..n} \longrightarrow \texttt{case } t_0' \texttt{ of <}l_i\texttt{=}x_i\texttt{>}\Rightarrow t_i\ ^{i\in 1..n}} \qquad \text{(E-Case)}$$

$$\frac{t_i \longrightarrow t_i'}{\texttt{<}l_i\texttt{=}t_i\texttt{> as T} \longrightarrow \texttt{<}l_i\texttt{=}t_i'\texttt{> as T}} \qquad \text{(E-Variant)}$$

New typing rules

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_j : T_j}{\Gamma \vdash \texttt{<}l_j\texttt{=}t_j\texttt{> as <}l_i\texttt{:}T_i\ ^{i\in 1..n}\texttt{> : <}l_i\texttt{:}T_i\ ^{i\in 1..n}\texttt{>}} \qquad \text{(T-Variant)}$$

$$\frac{\Gamma \vdash t_0 : \texttt{<}l_i\texttt{:}T_i\ ^{i\in 1..n}\texttt{>} \quad \text{for each } i \quad \Gamma, x_i\texttt{:}T_i \vdash t_i : T}{\Gamma \vdash \texttt{case } t_0 \texttt{ of <}l_i\texttt{=}x_i\texttt{>}\Rightarrow t_i\ ^{i\in 1..n} : T} \qquad \text{(T-Case)}$$

Example

```
Addr = <physical:PhysicalAddr, virtual:VirtualAddr>;

a = <physical=pa> as Addr;

getName = λa:Addr.
  case a of
    <physical=x> ⇒ x.firstlast
  | <virtual=y> ⇒ y.name;
```

Options

Just like in OCaml...

```
OptionalNat = <none:Unit, some:Nat>;

Table = Nat→OptionalNat;

emptyTable = λn:Nat. <none=unit> as OptionalNat;

extendTable =
  λt:Table. λm:Nat. λv:Nat.
    λn:Nat.
      if equal n m then <some=v> as OptionalNat
      else t n;

x = case t(5) of
      <none=u> ⇒ 999
    | <some=v> ⇒ v;
```

Enumerations

```
Weekday = <monday:Unit, tuesday:Unit, wednesday:Unit,
           thursday:Unit, friday:Unit>;

nextBusinessDay = λw:Weekday.
  case w of <monday=x>    ⇒ <tuesday=unit> as Weekday
          | <tuesday=x>   ⇒ <wednesday=unit> as Weekday
          | <wednesday=x> ⇒ <thursday=unit> as Weekday
          | <thursday=x>  ⇒ <friday=unit> as Weekday
          | <friday=x>    ⇒ <monday=unit> as Weekday;
```