Type Systems
Winter Semester 2006

Week 4

November 8

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The Lambda Calculus

The lambda-calculus

- If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
  - Turing complete
  - higher order (functions as data)
- Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- The *e. coli* of programming language research
- The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)

Intuitions

Suppose we want to describe a function that adds three to any number we pass it. We might write

$$\texttt{plus3 x} \quad = \quad \texttt{succ (succ (succ x))}$$

That is, "`plus3 x` is `succ (succ (succ x))`."

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A: `plus3` is the function that, given `x`, yields `succ (succ (succ x))`.

`plus3  =  λx. succ (succ (succ x))`

This function exists independent of the name `plus3`.

`λx. t` is written "`fun x → t`" in OCaml and "`x ⇒ t`" in Scala.

So `plus3 (succ 0)` is just a convenient shorthand for "the function that, given `x`, yields `succ (succ (succ x))`, applied to `succ 0`."

`plus3 (succ 0)`

=

`(λx. succ (succ (succ x))) (succ 0)`

Abstractions over Functions

Consider the λ-abstraction

```
g  =  λf. f (f (succ 0))
```

Note that the parameter variable `f` is used in the *function* position in the body of `g`. Terms like `g` are called *higher-order* functions. If we apply `g` to an argument like `plus3`, the "substitution rule" yields a nontrivial computation:

```
g plus3
  =   (λf. f (f (succ 0))) (λx. succ (succ (succ x)))
  i.e. (λx. succ (succ (succ x)))
         ((λx. succ (succ (succ x))) (succ 0))
  i.e. (λx. succ (succ (succ x)))
         (succ (succ (succ (succ 0))))
  i.e. succ (succ (succ (succ (succ (succ (succ 0)))))))
```

Abstractions Returning Functions

Consider the following variant of `g`:

```
double  =  λf. λy. f (f y)
```

I.e., `double` is the function that, when applied to a function `f`, yields a *function* that, when applied to an argument `y`, yields `f (f y)`.

Example

```
      double plus3 0
  =   (λf. λy. f (f y))
        (λx. succ (succ (succ x)))
        0
  i.e. (λy. (λx. succ (succ (succ x)))
              ((λx. succ (succ (succ x))) y))
        0
  i.e. (λx. succ (succ (succ x)))
              ((λx. succ (succ (succ x))) 0)
  i.e. (λx. succ (succ (succ x)))
              (succ (succ (succ 0)))
  i.e. succ (succ (succ (succ (succ (succ 0))))))
```

The Pure Lambda-Calculus

As the preceding examples suggest, once we have λ-abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the "pure lambda-calculus"— *everything* is a function.

- Variables always denote functions
- Functions always take other functions as parameters
- The result of a function is always a function

Formalities

Syntax

t ::=		*terms*
	x	*variable*
	λx.t	*abstraction*
	t t	*application*

*Term*inology:

- terms in the pure λ-calculus are often called λ-*terms*
- terms of the form λx. t are called λ-*abstractions* or just *abstractions*

Syntactic conventions

Since λ-calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- Application associates to the left

  E.g., $t\ u\ v$ *means* $(t\ u)\ v$, *not* $t\ (u\ v)$

- Bodies of λ- abstractions extend as far to the right as possible

  E.g., $\lambda x.\ \lambda y.\ x\ y$ *means* $\lambda x.\ (\lambda y.\ x\ y)$, *not* $\lambda x.\ (\lambda y.\ x)\ y$

Scope

The λ-abstraction term λx.t *binds* the variable x.

The *scope* of this binding is the *body* t.

Occurrences of x inside t are said to be *bound* by the abstraction.

Occurrences of x that are *not* within the scope of an abstraction binding x are said to be *free*.

Test:

$$\lambda x.\ \lambda y.\ x\ y\ z$$

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$$\lambda x.\ (\lambda y.\ z\ y)\ y$$

Values

v ::=		*values*
	$\lambda x.t$	*abstraction value*

Operational Semantics

Computation rule:

$$(\lambda x.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \qquad \text{(E-AppAbs)}$$

Notation: $[x \mapsto v_2]t_{12}$ *is "the term that results from substituting free occurrences of* x *in* t_{12} *with* v_{12}*."*

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Congruence rules:

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'} \qquad \text{(E-App2)}$$

Terminology

A term of the form `(`λ`x.t) v` — that is, a λ-abstraction applied to a *value* — is called a *redex* (short for "reducible expression").

Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*.

The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages.

Some other common ones:

- Call by name (cf. Haskell)
- Normal order (leftmost/outermost)
- Full (non-deterministic) beta-reduction

Classical Lambda Calculus

Full beta reduction

The classical lambda calculus allows full beta reduction.

- The argument of a β-reduction to be an arbitrary term, not just a value.
- Reduction may appear anywhere in a term.

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$$\frac{t_2 \longrightarrow t_2'}{t_1\ t_2 \longrightarrow t_1\ t_2'} \qquad \text{(E-App2)}$$

$$\frac{t \longrightarrow t'}{\lambda x.t \longrightarrow \lambda x.t'} \qquad \text{(E-Abs)}$$

Substitution revisited

> *Remember:* $[x \mapsto v_2]t_{12}$ *is "the term that results from substituting free occurrences of* x *in* t_{12} *with* v_{12}*."*

This is trickier than it looks!
For example:

$$\begin{array}{ll} & (\lambda x.\ (\lambda y.\ x))\ y \\ \longrightarrow & [x \mapsto y]\lambda y.\ x \\ = & ??? \end{array}$$

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Solution:
need to rename bound variables before performing the substitution.

$$\begin{array}{ll} & (\lambda x.\ (\lambda y.\ x))\ y \\ = & (\lambda x.\ (\lambda z.\ x))\ y \\ \longrightarrow & [x \mapsto y]\lambda z.\ x \\ = & \lambda z.\ y \end{array}$$

Alpha conversion

Renaming bound variables is formalized as α-conversion.
Conversion rule:

$$\frac{\mathtt{y} \notin \mathtt{fv(t)}}{\lambda\mathtt{x.\ t} =_\alpha \lambda\mathtt{y.[x} \mapsto \mathtt{y]t}} \qquad (\alpha)$$

Equivalence rules:

$$\frac{\mathtt{t}_1 =_\alpha \mathtt{t}_2}{\mathtt{t}_2 =_\alpha \mathtt{t}_1} \qquad (\alpha\text{-}\textsc{Symm})$$

$$\frac{\mathtt{t}_1 =_\alpha \mathtt{t}_2 \qquad \mathtt{t}_2 =_\alpha \mathtt{t}_3}{\mathtt{t}_1 =_\alpha \mathtt{t}_3} \qquad (\alpha\text{-}\textsc{Trans})$$

Congruence rules: the usual ones.

Confluence

Full β-reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

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Full β-reduction makes it possible to have different reduction paths.

Q: Can a term evaluate to more than one normal form?

The answer is no; this is a consequence of the following

Theorem [Church-Rosser]
Let $\mathtt{t}$, $\mathtt{t}_1$, $\mathtt{t}_2$ be terms such that $\mathtt{t} \longrightarrow^* \mathtt{t}_1$ and $\mathtt{t} \longrightarrow^* \mathtt{t}_2$. Then there exists a term $\mathtt{t}_3$ such that $\mathtt{t}_1 \longrightarrow^* \mathtt{t}_3$ and $\mathtt{t}_2 \longrightarrow^* \mathtt{t}_3$.

Programming in the Lambda-Calculus

Multiple arguments

Consider the function `double`, which returns a function as an argument.

$$\texttt{double} \;=\; \lambda\texttt{f. }\lambda\texttt{y. f (f y)}$$

This idiom — a λ-abstraction that does nothing but immediately yield another abstraction — is very common in the λ-calculus.

In general, λ`x.` λ`y. t` is a function that, given a value `v` for `x`, yields a function that, given a value `u` for `y`, yields `t` with `v` in place of `x` and `u` in place of `y`.

That is, λ`x.` λ`y. t` is a two-argument function.

(Recall the discussion of *currying* in OCaml.)

The "Church Booleans"

$$\begin{array}{lcl}\texttt{tru} & = & \lambda\texttt{t. }\lambda\texttt{f. t}\\ \texttt{fls} & = & \lambda\texttt{t. }\lambda\texttt{f. f}\end{array}$$

$$\begin{array}{lll} & \texttt{tru v w} & \\ = & \underline{(\lambda\texttt{t.}\lambda\texttt{f.t})\ \texttt{v}}\ \texttt{w} & \text{by definition}\\ \longrightarrow & \underline{(\lambda\texttt{f. v})\ \texttt{w}} & \text{reducing the underlined redex}\\ \longrightarrow & \texttt{v} & \text{reducing the underlined redex}\end{array}$$

$$\begin{array}{lll} & \texttt{fls v w} & \\ = & \underline{(\lambda\texttt{t.}\lambda\texttt{f.f})\ \texttt{v}}\ \texttt{w} & \text{by definition}\\ \longrightarrow & \underline{(\lambda\texttt{f. f})\ \texttt{w}} & \text{reducing the underlined redex}\\ \longrightarrow & \texttt{w} & \text{reducing the underlined redex}\end{array}$$

Functions on Booleans

$$\texttt{not} \;=\; \lambda\texttt{b. b fls tru}$$

That is, `not` is a function that, given a boolean value `v`, returns `fls` if `v` is `tru` and `tru` if `v` is `fls`.

Functions on Booleans

$$\texttt{and} \;=\; \lambda\texttt{b. }\lambda\texttt{c. b c fls}$$

That is, `and` is a function that, given two boolean values `v` and `w`, returns `w` if `v` is `tru` and `fls` if `v` is `fls`
Thus `and v w` yields `tru` if both `v` and `w` are `tru` and `fls` if either `v` or `w` is `fls`.

Pairs

```
pair = λf.λs.λb. b f s
fst = λp. p tru
snd = λp. p fls
```

That is, `pair v w` is a function that, when applied to a boolean value `b`, applies `b` to `v` and `w`.
By the definition of booleans, this application yields `v` if `b` is `tru` and `w` if `b` is `fls`, so the first and second projection functions `fst` and `snd` can be implemented simply by supplying the appropriate boolean.

Example

	fst (pair v w)	
$=$	fst ((λf. λs. λb. b f s) v w)	by definition
$\longrightarrow$	fst ((λs. λb. b v s) w)	reducing
$\longrightarrow$	fst (λb. b v w)	reducing
$=$	(λp. p tru) (λb. b v w)	by definition
$\longrightarrow$	(λb. b v w) tru	reducing
$\longrightarrow$	tru v w	reducing
$\longrightarrow^*$	v	as before.

Church numerals

Idea: represent the number n by a function that "repeats some action n times."

c_0 = λs. λz. z
c_1 = λs. λz. s z
c_2 = λs. λz. s (s z)
c_3 = λs. λz. s (s (s z))

That is, each number n is represented by a term c_n that takes two arguments, `s` and `z` (for "successor" and "zero"), and applies `s`, n times, to `z`.

Functions on Church Numerals

Successor:

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Addition:

Functions on Church Numerals

Successor:

```
scc = λn. λs. λz. s (n s z)
```

Addition:

```
plus = λm. λn. λs. λz. m s (n s z)
```

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Successor:

```
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Multiplication:

Functions on Church Numerals

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Multiplication:

times = λm. λn. m (plus n) c_0

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Successor:

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Addition:

plus = λm. λn. λs. λz. m s (n s z)

Multiplication:

times = λm. λn. m (plus n) c_0

Zero test:

Functions on Church Numerals

Successor:

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Addition:

plus = λm. λn. λs. λz. m s (n s z)

Multiplication:

times = λm. λn. m (plus n) c_0

Zero test:

iszro = λm. m (λx. fls) tru

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times = λm. λn. m (plus n) c_0

Zero test:

iszro = λm. m (λx. fls) tru

What about predecessor?

Predecessor

zz = pair c_0 c_0

ss = λp. pair (snd p) (scc (snd p))

prd = λm. fst (m ss zz)

Normal forms

Recall:

- A *normal form* is a term that cannot take an evaluation step.
- A *stuck* term is a normal form that is not a value.

Are there any stuck terms in the pure λ-calculus?

Normal forms

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Are there any stuck terms in the pure λ-calculus?

Does every term evaluate to a normal form?

Divergence

$$\texttt{omega} \quad = \quad (\lambda\texttt{x. x x})\ (\lambda\texttt{x. x x})$$

Note that omega evaluates in one step to itself!
So evaluation of omega never reaches a normal form: it *diverges*.

Divergence

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So evaluation of omega never reaches a normal form: it *diverges*.

Being able to write a divergent computation does not seem very useful in itself. However, there are variants of omega that are *very* useful...

Recursion in the Lambda-Calculus

Iterated Application

Suppose f is some λ-abstraction, and consider the following term:

$$\texttt{Y}_f = (\lambda\texttt{x. f (x x)})\ (\lambda\texttt{x. f (x x)})$$

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Now the "pattern of divergence" becomes more interesting:

$$\begin{array}{c}
\texttt{Y}_f \\
= \\
\underline{(\lambda\texttt{x. f (x x)})\ (\lambda\texttt{x. f (x x)})} \\
\longrightarrow \\
\texttt{f (}\underline{(\lambda\texttt{x. f (x x)})\ (\lambda\texttt{x. f (x x)})}\texttt{)} \\
\longrightarrow \\
\texttt{f (f (}\underline{(\lambda\texttt{x. f (x x)})\ (\lambda\texttt{x. f (x x)})}\texttt{))} \\
\longrightarrow \\
\texttt{f (f (f (}\underline{(\lambda\texttt{x. f (x x)})\ (\lambda\texttt{x. f (x x)})}\texttt{)))} \\
\longrightarrow \\
\ldots
\end{array}$$

$\mathtt{Y}_f$ is still not very useful, since (like `omega`), all it does is diverge. Is there any way we could "slow it down"?

Delaying divergence

$$\texttt{poisonpill} \quad = \quad \lambda\texttt{y. omega}$$

Note that `poisonpill` is a value — it it will only diverge when we actually apply it to an argument. This means that we can safely pass it as an argument to other functions, return it as a result from functions, etc.

$$\begin{array}{c} \underline{\texttt{(}\lambda\texttt{p. fst (pair p fls) tru) poisonpill}} \\ \longrightarrow \\ \texttt{fst (pair poisonpill fls) tru} \\ \longrightarrow^* \\ \underline{\texttt{poisonpill tru}} \\ \longrightarrow \\ \texttt{omega} \\ \longrightarrow \\ \cdots \end{array}$$

A delayed variant of omega

Here is a variant of `omega` in which the delay and divergence are a bit more tightly intertwined:

$$\begin{array}{c} \texttt{omegav} \quad = \\ \lambda\texttt{y. (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) y} \end{array}$$

Note that `omegav` is a normal form. However, if we apply it to any argument `v`, it diverges:

$$\begin{array}{c} \texttt{omegav v} \\ = \\ \underline{\texttt{(}\lambda\texttt{y. (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) y) v}} \\ \longrightarrow \\ \underline{\texttt{(}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) (}\lambda\texttt{x. (}\lambda\texttt{y. x x y))}}\texttt{ v} \\ \longrightarrow \\ \texttt{(}\lambda\texttt{y. (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) (}\lambda\texttt{x. (}\lambda\texttt{y. x x y)) y) v} \\ = \\ \texttt{omegav v} \end{array}$$

Another delayed variant

Suppose `f` is a function. Define

$$\mathtt{Z}_f \quad = \quad \lambda\texttt{y. (}\lambda\texttt{x. f (}\lambda\texttt{y. x x y)) (}\lambda\texttt{x. f (}\lambda\texttt{y. x x y)) y}$$

This term combines the "added `f`" from $\mathtt{Y}_f$ with the "delayed divergence" of `omegav`.

If we now apply Z_f to an argument v, something interesting happens:

$$
\begin{array}{c}
Z_f \; v \\
= \\
\underline{(\lambda y.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ y)\ v} \\
\longrightarrow \\
\underline{(\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))}\ v \\
\longrightarrow \\
f\ (\lambda y.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ y)\ v \\
= \\
f\ Z_f\ v
\end{array}
$$

Since Z_f and v are both values, the next computation step will be the reduction of f Z_f — that is, before we "diverge," f gets to do some computation.
Now we are getting somewhere.

Recursion

Let

```
f   =   λfct.
          λn.
            if n=0 then 1
            else n * (fct (pred n))
```

f looks just the ordinary factorial function, except that, in place of a recursive call in the last time, it calls the function fct, which is passed as a parameter.

N.b.: for brevity, this example uses "real" numbers and booleans, infix syntax, etc. It can easily be translated into the pure lambda-calculus (using Church numerals, etc.).

We can use Z to "tie the knot" in the definition of f and obtain a real recursive factorial function:

$$
\begin{array}{c}
Z_f\ 3 \\
\longrightarrow^* \\
f\ Z_f\ 3 \\
= \\
(\lambda fct.\ \lambda n.\ \ldots)\ Z_f\ 3 \\
\longrightarrow\ \longrightarrow \\
\text{if } 3{=}0 \text{ then } 1 \text{ else } 3 * (Z_f\ (\text{pred } 3)) \\
\longrightarrow^* \\
3 * (Z_f\ (\text{pred } 3))) \\
\longrightarrow \\
3 * (Z_f\ 2) \\
\longrightarrow^* \\
3 * (f\ Z_f\ 2) \\
\ldots
\end{array}
$$

A Generic Z

If we define

$$Z \quad = \quad \lambda f.\ Z_f$$

i.e.,

$$
\begin{array}{c}
Z \quad = \\
\lambda f.\ \lambda y.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ y
\end{array}
$$

then we can obtain the behavior of Z_f for any f we like, simply by applying Z to f.

$$Z\ f \quad \longrightarrow \quad Z_f$$

For example:

```
fact    =    Z  ( λfct.
                    λn.
                      if n=0 then 1
                      else n * (fct (pred n)) )
```

Technical Note

The term `Z` here is essentially the same as the `fix` discussed the book.

```
Z =
  λf. λy. (λx. f (λy. x x y)) (λx. f (λy. x x y)) y

fix =
  λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))
```

`Z` is hopefully slightly easier to understand, since it has the property that `Z f v` $\longrightarrow^*$ `f (Z f) v`, which `fix` does not (quite) share.