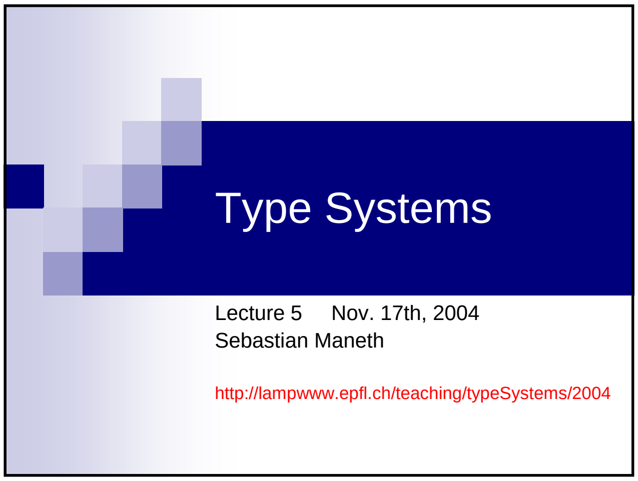
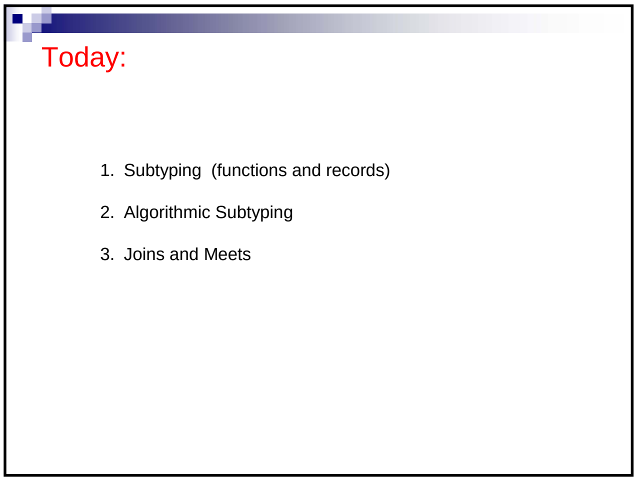


Type Systems

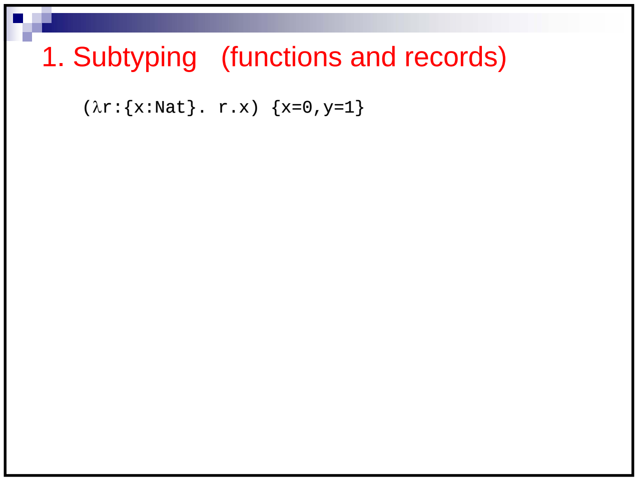
Lecture 5 Nov. 17th, 2004
Sebastian Maneth

http://lampwww.epfl.ch/teaching/typeSystems/2004



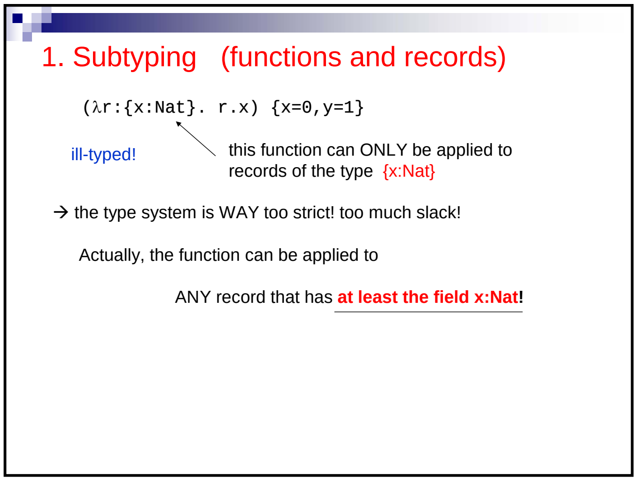
Today:

1. Subtyping (functions and records)
2. Algorithmic Subtyping
3. Joins and Meets



1. Subtyping (functions and records)

```
(λr:{x:Nat}. r.x) {x=0,y=1}
```



1. Subtyping (functions and records)

```
(λr:{x:Nat}. r.x) {x=0,y=1}
```

ill-typed!

this function can ONLY be applied to records of the type {x:Nat}

→ the type system is WAY too strict! too much slack!

Actually, the function can be applied to

ANY record that has **at least the field x:Nat!**

1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}`

ill-typed!

this function can ONLY be applied to records of the type {x:Nat}

→ the type system is WAY too strict! too much slack!

Actually, the function can be applied to

ANY record that has **at least the field x:Nat!**

a **subtype** of {x:Nat}

Why "**sub**"?

{x:Nat,y:Nat} is a subtype, written **<:** of {x:Nat} because

# of records having this is LESS than # of records having this

1. Subtyping (functions and records)

{x:Nat,y:Nat} **<:** {x:Nat} ("is subtype of")

1. Subtyping (functions and records)

{x:Nat,y:Nat} **<:** {x:Nat} ("is subtype of")

S T

→ If `t` satisfies S in some context, then it also satisfies T!!

1. Subtyping (functions and records)

{x:Nat,y:Nat} **<:** {x:Nat} ("is subtype of")

S T

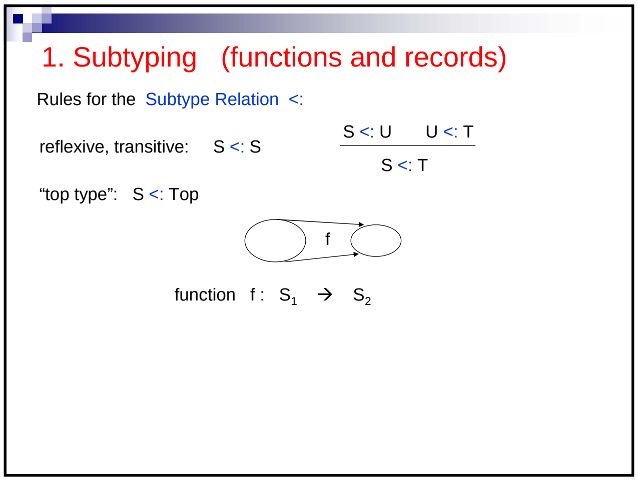
→ If `t` satisfies S in some context, then it also satisfies T!!

is of type is of type

Rule of subsumtion:

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

… = "move to a more general type (supertype)"



1. Subtyping (functions and records)

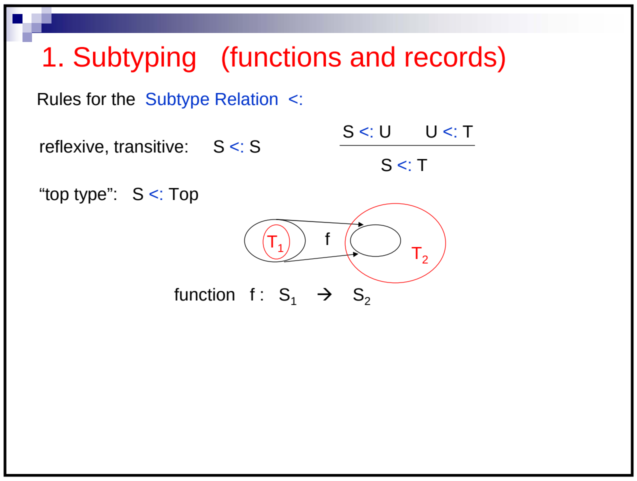
Rules for the Subtype Relation <:

reflexive, transitive: $S <: S$

$$\frac{S <: U \qquad U <: T}{S <: T}$$

"top type": $S <: Top$

function $f : S_1 \to S_2$



1. Subtyping (functions and records)

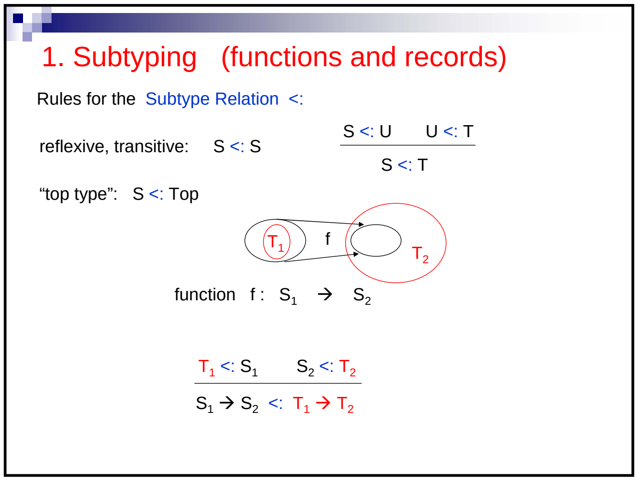
Rules for the Subtype Relation <:

reflexive, transitive: $S <: S$

$$\frac{S <: U \qquad U <: T}{S <: T}$$

"top type": $S <: Top$

function $f : S_1 \to S_2$



1. Subtyping (functions and records)

Rules for the Subtype Relation <:

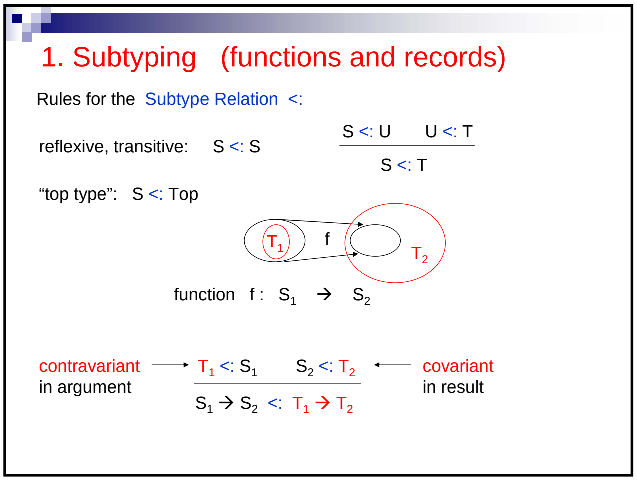
reflexive, transitive: $S <: S$

$$\frac{S <: U \qquad U <: T}{S <: T}$$

"top type": $S <: Top$

function $f : S_1 \to S_2$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$



1. Subtyping (functions and records)

Rules for the Subtype Relation <:

reflexive, transitive: $S <: S$

$$\frac{S <: U \qquad U <: T}{S <: T}$$

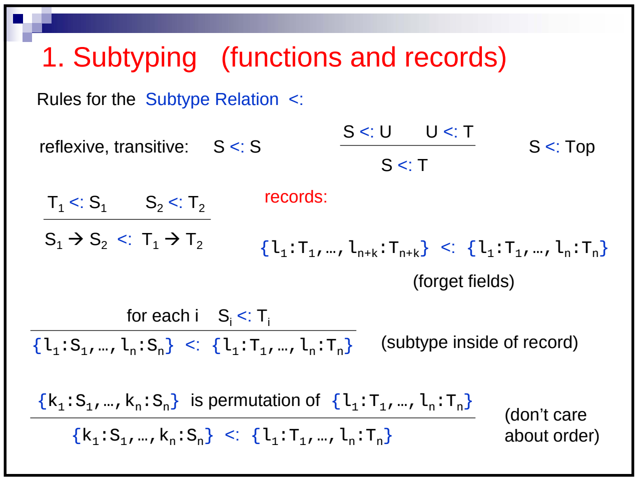
"top type": $S <: Top$

function $f : S_1 \to S_2$

contravariant in argument →

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$

← covariant in result



1. Subtyping (functions and records)

Rules for the Subtype Relation <:

reflexive, transitive: $S <: S$ $\qquad \dfrac{S <: U \qquad U <: T}{S <: T}$ $\qquad S <: Top$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$

records:

$$\{l_1:T_1,\dots,l_{n+k}:T_{n+k}\} <: \{l_1:T_1,\dots,l_n:T_n\}$$

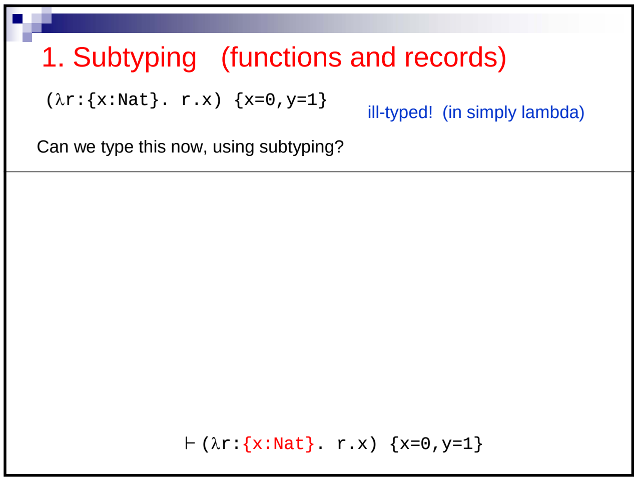
(forget fields)

$$\frac{\text{for each } i \quad S_i <: T_i}{\{l_1:S_1,\dots,l_n:S_n\} <: \{l_1:T_1,\dots,l_n:T_n\}}$$

(subtype inside of record)

$$\frac{\{k_1:S_1,\dots,k_n:S_n\} \text{ is permutation of } \{l_1:T_1,\dots,l_n:T_n\}}{\{k_1:S_1,\dots,k_n:S_n\} <: \{l_1:T_1,\dots,l_n:T_n\}}$$

(don't care about order)

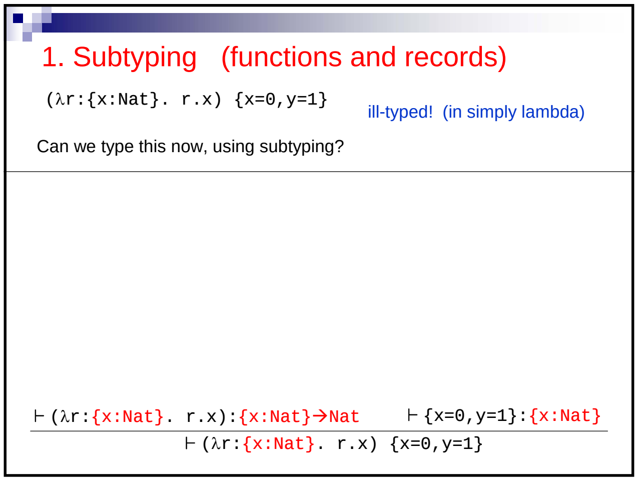


1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\vdash (\lambda r:\{x:Nat\}.\ r.x)\ \{x=0,y=1\}$$

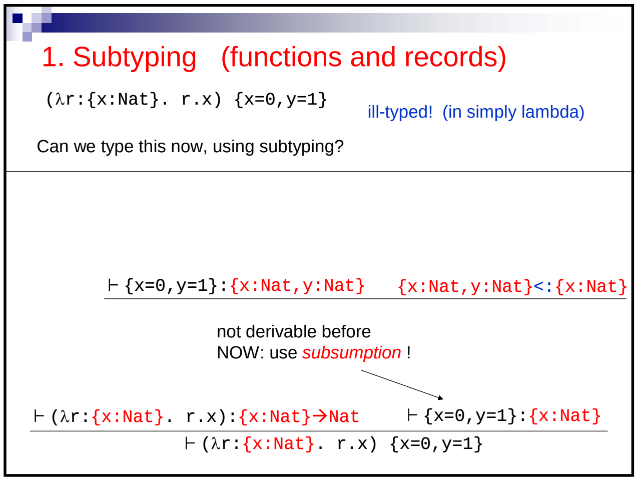


1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\frac{\vdash (\lambda r:\{x:Nat\}.\ r.x):\{x:Nat\}\to Nat \qquad \vdash \{x=0,y=1\}:\{x:Nat\}}{\vdash (\lambda r:\{x:Nat\}.\ r.x)\ \{x=0,y=1\}}$$



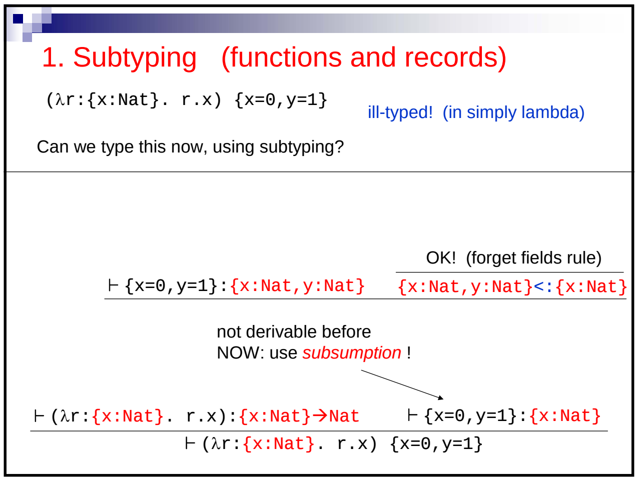
1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\frac{\vdash (\lambda r:\{x:Nat\}.\ r.x):\{x:Nat\}\to Nat \qquad \dfrac{\vdash \{x=0,y=1\}:\{x:Nat,y:Nat\} \qquad \{x:Nat,y:Nat\}<:\{x:Nat\}}{\vdash \{x=0,y=1\}:\{x:Nat\}}}{\vdash (\lambda r:\{x:Nat\}.\ r.x)\ \{x=0,y=1\}}$$

not derivable before
NOW: use *subsumption* !



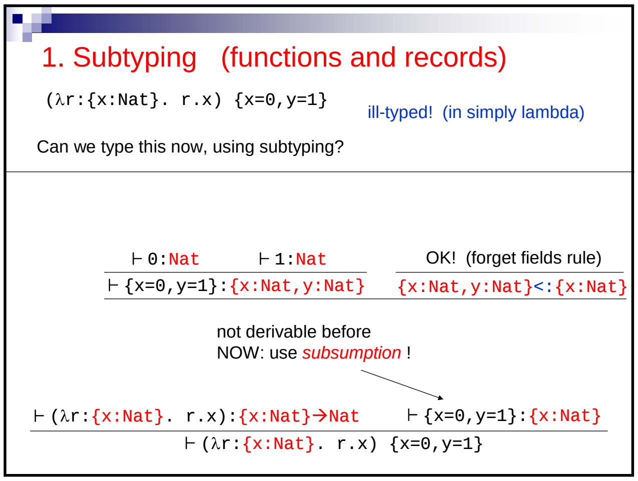
1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x):\{x{:}Nat\}\to Nat \qquad \dfrac{\dfrac{}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat,y{:}Nat\}} \qquad \dfrac{\text{OK! (forget fields rule)}}{\{x{:}Nat,y{:}Nat\}<:\{x{:}Nat\}}}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat\}}}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x)\ \{x{=}0,y{=}1\}}$$

not derivable before
NOW: use *subsumption* !



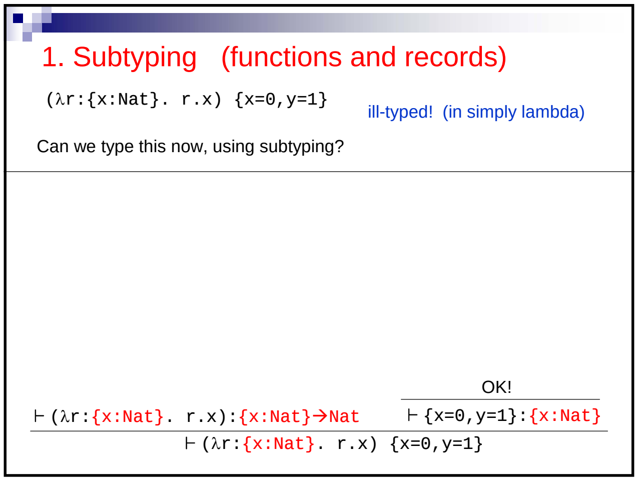
1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x):\{x{:}Nat\}\to Nat \qquad \dfrac{\dfrac{\vdash 0:Nat \qquad \vdash 1:Nat}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat,y{:}Nat\}} \qquad \dfrac{\text{OK! (forget fields rule)}}{\{x{:}Nat,y{:}Nat\}<:\{x{:}Nat\}}}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat\}}}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x)\ \{x{=}0,y{=}1\}}$$

not derivable before
NOW: use *subsumption* !

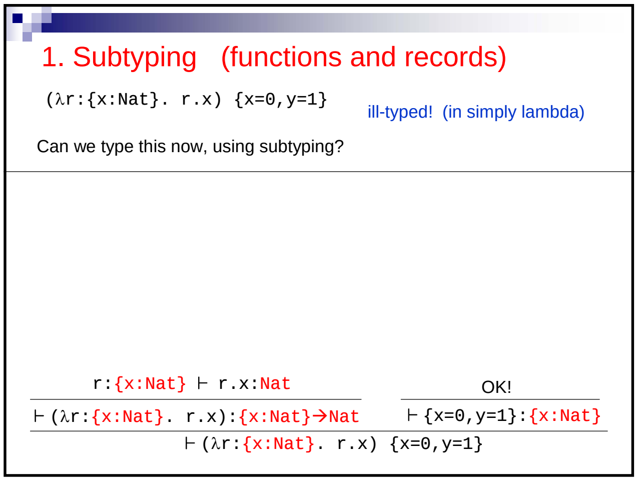


1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x):\{x{:}Nat\}\to Nat \qquad \dfrac{\text{OK!}}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat\}}}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x)\ \{x{=}0,y{=}1\}}$$



1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\dfrac{r{:}\{x{:}Nat\} \vdash r.x{:}Nat}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x):\{x{:}Nat\}\to Nat} \qquad \dfrac{\text{OK!}}{\vdash \{x{=}0,y{=}1\}:\{x{:}Nat\}}}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x)\ \{x{=}0,y{=}1\}}$$

1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\dfrac{\dfrac{r:\{x:Nat\} \vdash r:\{x:Nat\}}{r:\{x:Nat\} \vdash r.x:Nat}}{\vdash (\lambda r:\{x:Nat\}.\ r.x):\{x:Nat\}\to Nat} \qquad \dfrac{\text{OK!}}{\vdash \{x=0,y=1\}:\{x:Nat\}}}{\vdash (\lambda r:\{x:Nat\}.\ r.x)\ \{x=0,y=1\}}$$

1. Subtyping (functions and records)

`(λr:{x:Nat}. r.x) {x=0,y=1}` ill-typed! (in simply lambda)

Can we type this now, using subtyping?

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\text{OK!}\\ r:\{x:Nat\} \in r:\{x:Nat\}\end{matrix}}{r:\{x:Nat\} \vdash r:\{x:Nat\}}}{\dfrac{r:\{x:Nat\} \vdash r.x:Nat}{\vdash (\lambda r:\{x:Nat\}.\ r.x):\{x:Nat\}\to Nat}} \qquad \dfrac{\text{OK!}}{\vdash \{x=0,y=1\}:\{x:Nat\}}}{\vdash (\lambda r:\{x:Nat\}.\ r.x)\ \{x=0,y=1\}}$$

2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\frac{S <: U \qquad U <: T}{S <: T}$$

Are NOT syntax-directed!!

→ When to apply them??

$$S <: S$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\frac{S <: U \qquad U <: T}{S <: T}$$

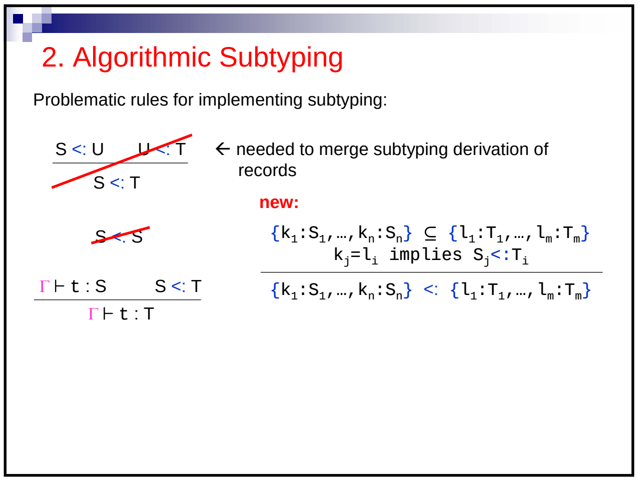
← needed to merge subtyping derivation of records

to see this, find a derivation for

`{x:{a:Nat,b:Nat},y:{m:Nat}}`
`<:{x:{a:Nat}}`

$$S <: S$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$



2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\cancel{\frac{S <: U \qquad U <: T}{S <: T}}$$

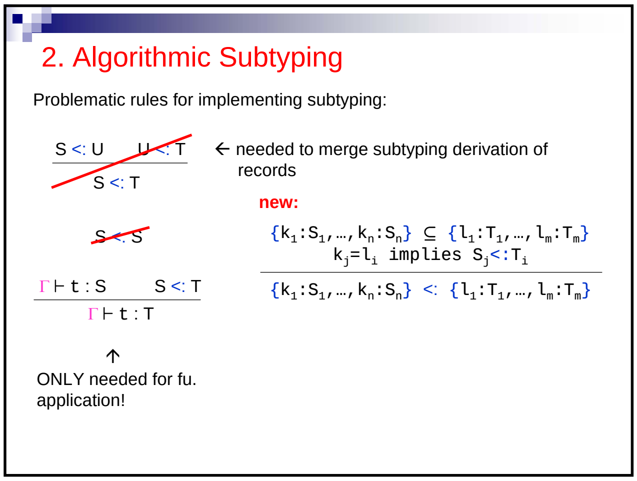
← needed to merge subtyping derivation of records

$$\cancel{S <: S}$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

new:

$$\frac{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\ldots,l_m{:}T_m\} \qquad k_j=l_i \text{ implies } S_j<:T_i}{\{k_1{:}S_1,\ldots,k_n{:}S_n\} <: \{l_1{:}T_1,\ldots,l_m{:}T_m\}}$$



2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\cancel{\frac{S <: U \qquad U <: T}{S <: T}}$$

← needed to merge subtyping derivation of records

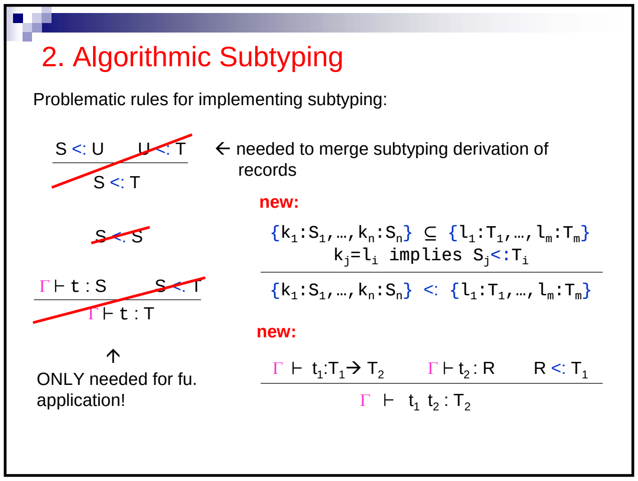
$$\cancel{S <: S}$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

↑
ONLY needed for fu. application!

new:

$$\frac{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\ldots,l_m{:}T_m\} \qquad k_j=l_i \text{ implies } S_j<:T_i}{\{k_1{:}S_1,\ldots,k_n{:}S_n\} <: \{l_1{:}T_1,\ldots,l_m{:}T_m\}}$$



2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\cancel{\frac{S <: U \qquad U <: T}{S <: T}}$$

← needed to merge subtyping derivation of records

$$\cancel{S <: S}$$

$$\cancel{\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}}$$

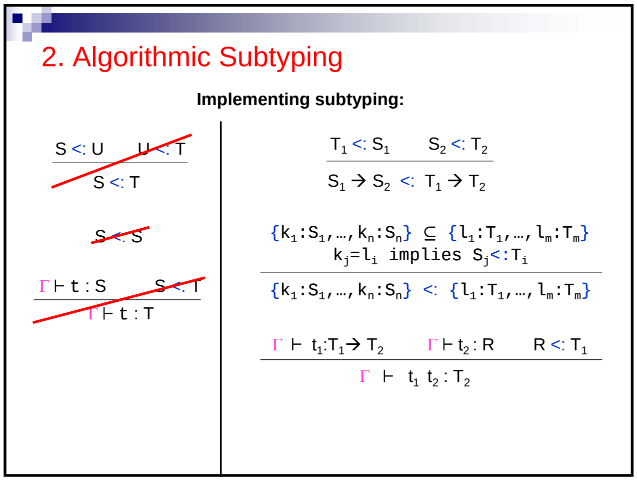
↑
ONLY needed for fu. application!

new:

$$\frac{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\ldots,l_m{:}T_m\} \qquad k_j=l_i \text{ implies } S_j<:T_i}{\{k_1{:}S_1,\ldots,k_n{:}S_n\} <: \{l_1{:}T_1,\ldots,l_m{:}T_m\}}$$

new:

$$\frac{\Gamma \vdash t_1{:}T_1 \to T_2 \qquad \Gamma \vdash t_2 : R \qquad R <: T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$



2. Algorithmic Subtyping

Implementing subtyping:

$$\cancel{\frac{S <: U \qquad U <: T}{S <: T}}$$

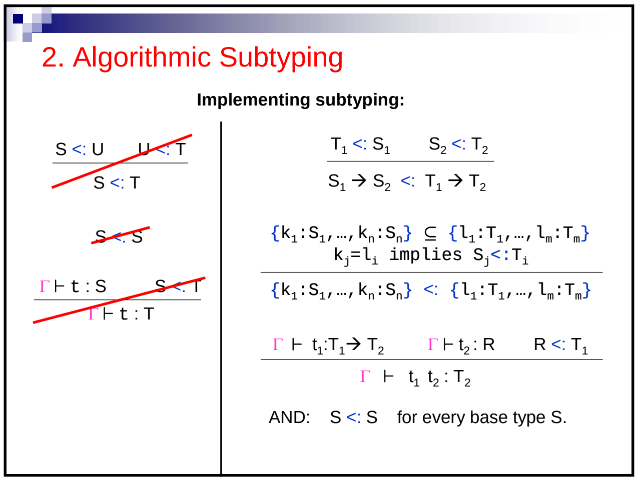
$$\cancel{S <: S}$$

$$\cancel{\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}}$$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$

$$\frac{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\ldots,l_m{:}T_m\} \qquad k_j=l_i \text{ implies } S_j<:T_i}{\{k_1{:}S_1,\ldots,k_n{:}S_n\} <: \{l_1{:}T_1,\ldots,l_m{:}T_m\}}$$

$$\frac{\Gamma \vdash t_1{:}T_1 \to T_2 \qquad \Gamma \vdash t_2 : R \qquad R <: T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$



2. Algorithmic Subtyping

Implementing subtyping:

$$\cancel{\dfrac{S <: U \qquad U <: T}{S <: T}}$$

$$\cancel{S <: S}$$

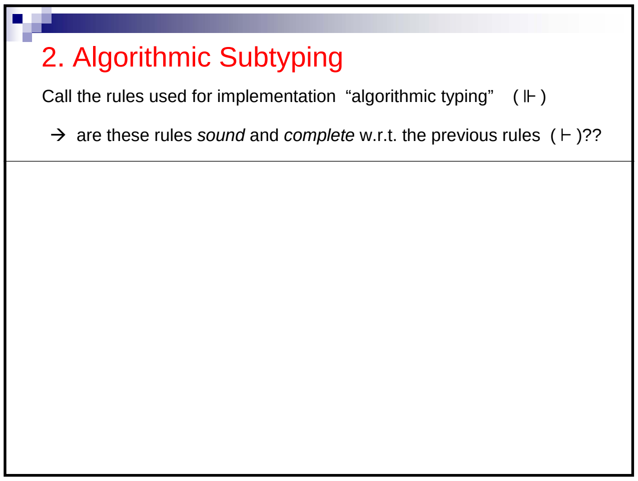
$$\cancel{\dfrac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}}$$

$$\dfrac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \rightarrow S_2 \;<:\; T_1 \rightarrow T_2}$$

$$\dfrac{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\ldots,l_m{:}T_m\} \quad k_j = l_i \text{ implies } S_j <: T_i}{\{k_1{:}S_1,\ldots,k_n{:}S_n\} \;<:\; \{l_1{:}T_1,\ldots,l_m{:}T_m\}}$$

$$\dfrac{\Gamma \vdash t_1{:}T_1 \rightarrow T_2 \qquad \Gamma \vdash t_2 : R \qquad R <: T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

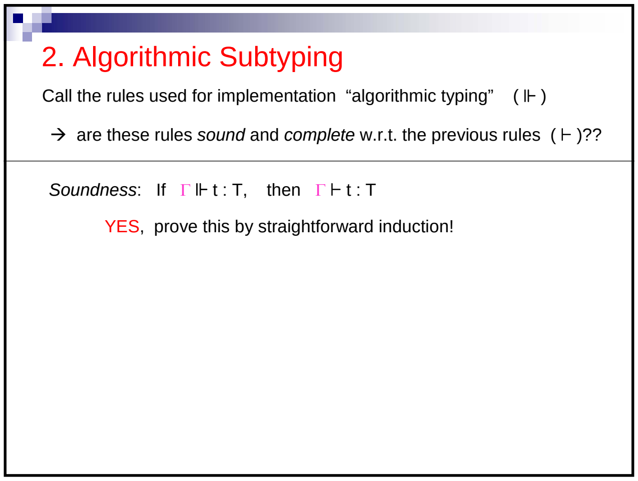
AND: $S <: S$ for every base type S.



2. Algorithmic Subtyping

Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??



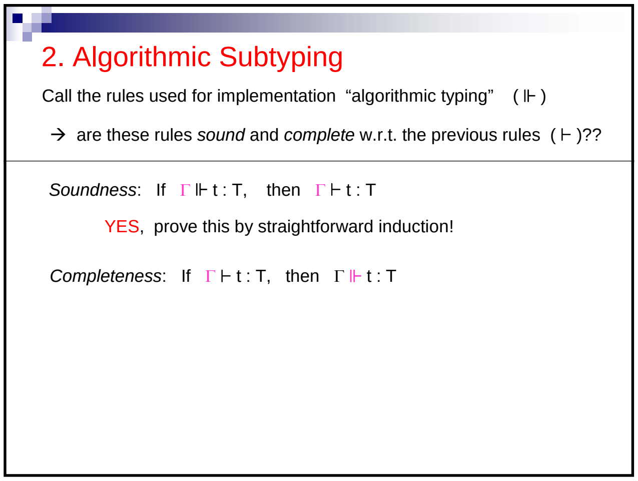
2. Algorithmic Subtyping

Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

Soundness: If $\Gamma \Vdash t : T$, then $\Gamma \vdash t : T$

YES, prove this by straightforward induction!



2. Algorithmic Subtyping

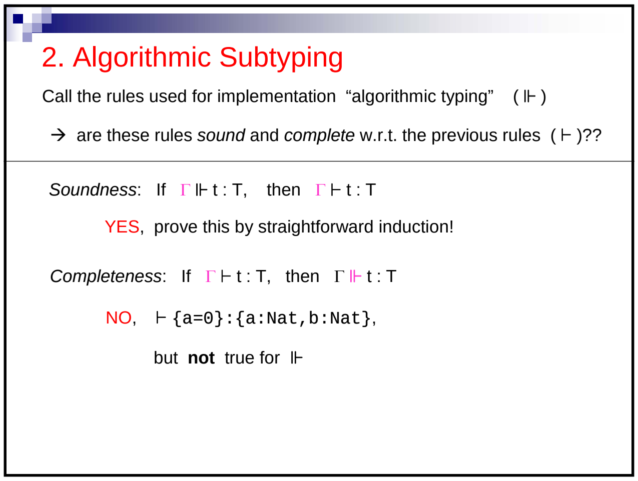
Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

Soundness: If $\Gamma \Vdash t : T$, then $\Gamma \vdash t : T$

YES, prove this by straightforward induction!

Completeness: If $\Gamma \vdash t : T$, then $\Gamma \Vdash t : T$



2. Algorithmic Subtyping

Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

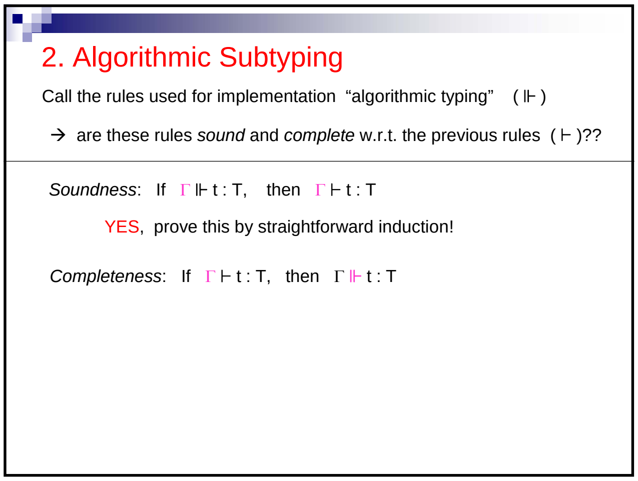
Soundness: If $\Gamma \Vdash t : T$, then $\Gamma \vdash t : T$

YES, prove this by straightforward induction!

Completeness: If $\Gamma \vdash t : T$, then $\Gamma \Vdash t : T$

NO, $\vdash$ `{a=0}:{a:Nat,b:Nat}`,

but **not** true for $\Vdash$



2. Algorithmic Subtyping

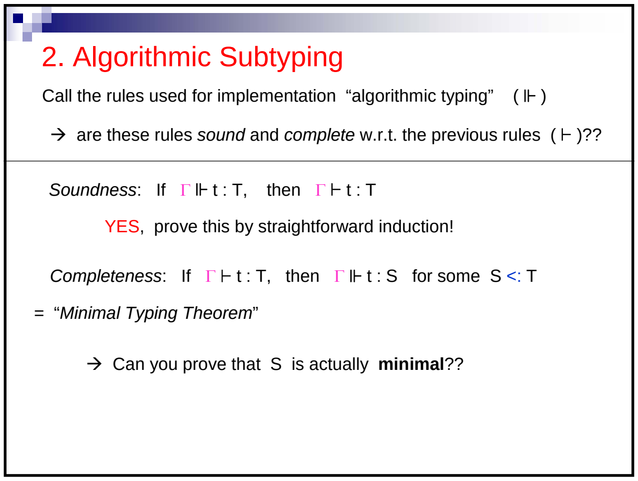
Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

Soundness: If $\Gamma \Vdash t : T$, then $\Gamma \vdash t : T$

YES, prove this by straightforward induction!

Completeness: If $\Gamma \vdash t : T$, then $\Gamma \Vdash t : T$



2. Algorithmic Subtyping

Call the rules used for implementation "algorithmic typing" ($\Vdash$)

→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

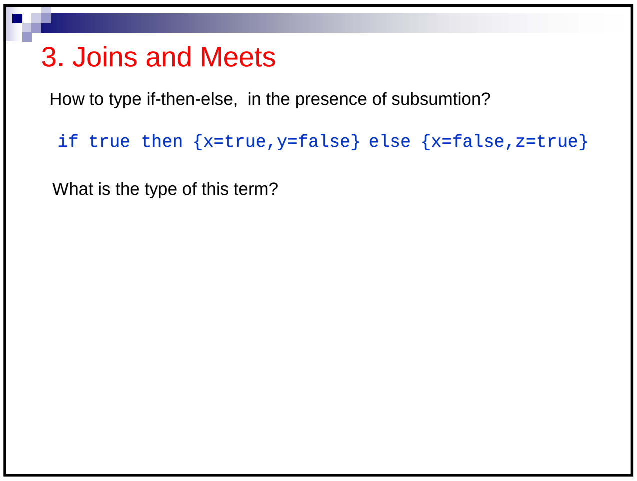
Soundness: If $\Gamma \Vdash t : T$, then $\Gamma \vdash t : T$

YES, prove this by straightforward induction!

Completeness: If $\Gamma \vdash t : T$, then $\Gamma \Vdash t : S$ for some $S <: T$

= "*Minimal Typing Theorem*"

→ Can you prove that S is actually **minimal**??

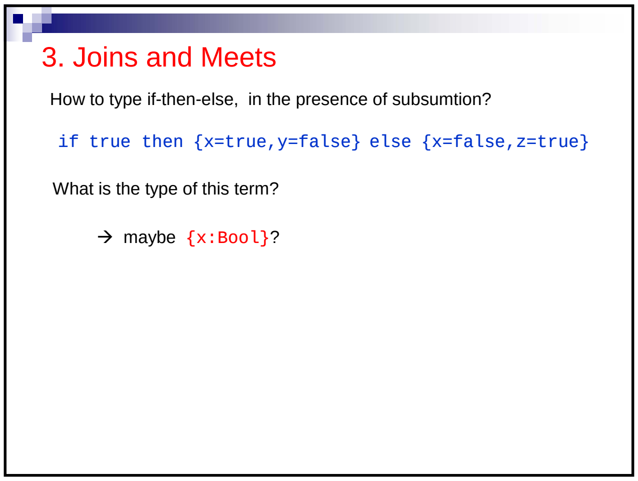


3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?



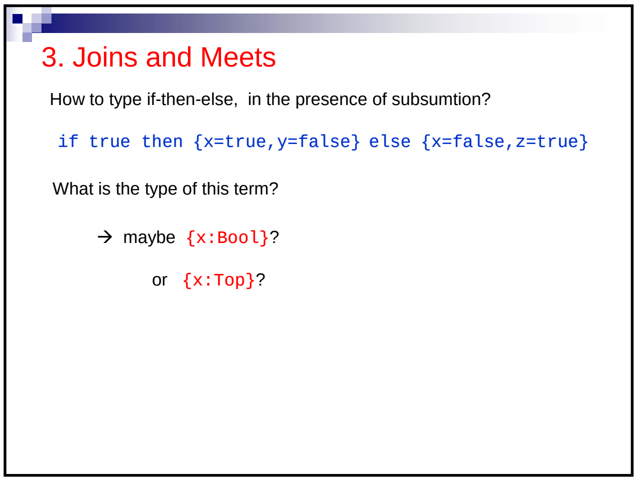
3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ maybe `{x:Bool}`?



3. Joins and Meets

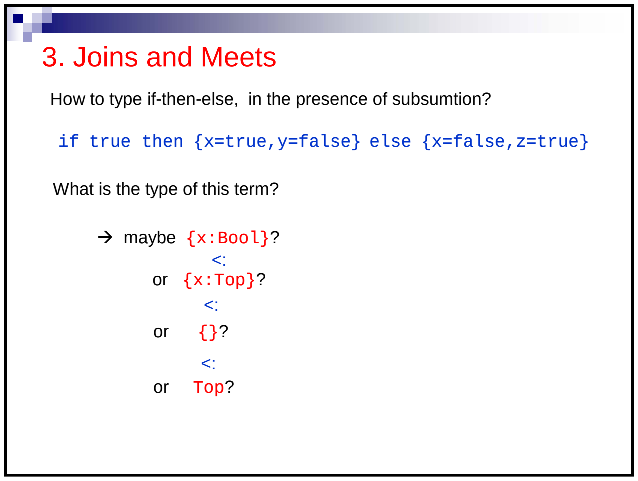
How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ maybe `{x:Bool}`?

or `{x:Top}`?



3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ maybe `{x:Bool}`?

<:

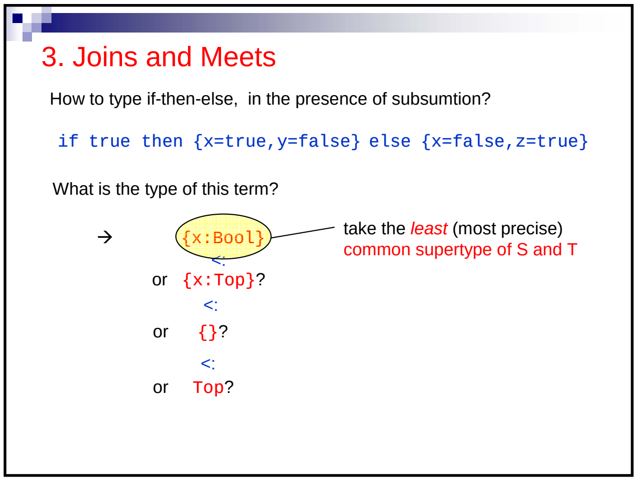
or `{x:Top}`?

<:

or `{}`?

<:

or `Top`?



3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ `{x:Bool}`

take the *least* (most precise) common supertype of S and T

<:

or `{x:Top}`?

<:

or `{}`?

<:

or `Top`?

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ {x:Bool}

take the *least* (most precise) common supertype of S and T

<:

or {x:Top}?

<:

or {}?

<:

or Top?

= "the *join* of S and T"

=: S ∨ T

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

$$\frac{t_1 : \mathrm{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T}$$

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

$$\frac{t_1 : \mathrm{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T}$$

$S \vee T$:= Bool, if S = T = Bool

$\{j_1:J_1,\ldots,j_q:J_q\}$ if $S = \{k_1:S_1,\ldots,k_m:S_m\}$,
$T = \{l_1:T_1,\ldots,l_n:T_n\}$,
$\{j_1:J_1,\ldots,j_q:J_q\} = S \cap T$, and
$J_u = S_v \vee T_w$ for $j_u = k_v = l_w$

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

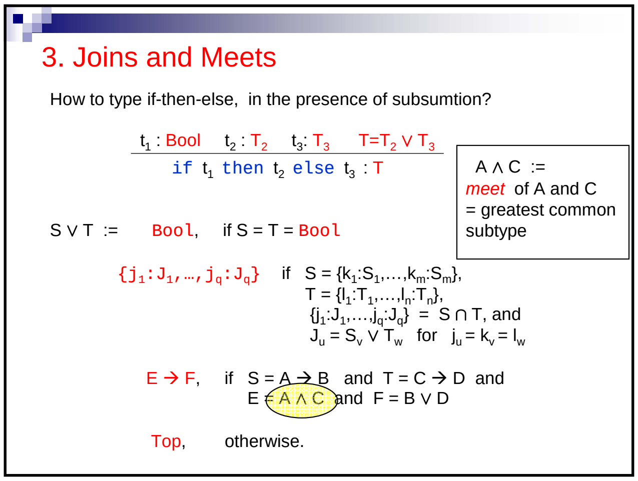
$$\frac{t_1 : \mathrm{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T}$$

$A \wedge C$:=
meet of A and C
= greatest common subtype

$S \vee T$:= Bool, if S = T = Bool

$\{j_1:J_1,\ldots,j_q:J_q\}$ if $S = \{k_1:S_1,\ldots,k_m:S_m\}$,
$T = \{l_1:T_1,\ldots,l_n:T_n\}$,
$\{j_1:J_1,\ldots,j_q:J_q\} = S \cap T$, and
$J_u = S_v \vee T_w$ for $j_u = k_v = l_w$

$E \to F$, if $S = A \to B$ and $T = C \to D$ and
$E = A \wedge C$ and $F = B \vee D$



3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

$$\frac{t_1 : \mathrm{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T}$$

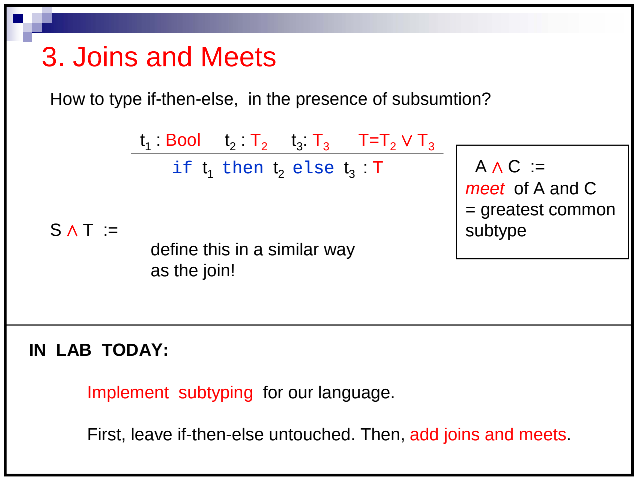
$A \wedge C$:=
meet of A and C
= greatest common subtype

$S \vee T$:= $\texttt{Bool}$, if $S = T = \texttt{Bool}$

$\{j_1{:}J_1, \dots, j_q{:}J_q\}$ if $S = \{k_1{:}S_1, \dots, k_m{:}S_m\}$,
$T = \{l_1{:}T_1, \dots, l_n{:}T_n\}$,
$\{j_1{:}J_1, \dots, j_q{:}J_q\} = S \cap T$, and
$J_u = S_v \vee T_w$ for $j_u = k_v = l_w$

$E \to F$, if $S = A \to B$ and $T = C \to D$ and
$E = A \wedge C$ and $F = B \vee D$

Top, otherwise.



3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

$$\frac{t_1 : \mathrm{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3 : T}$$

$A \wedge C$:=
meet of A and C
= greatest common subtype

$S \wedge T$:=

define this in a similar way
as the join!

IN LAB TODAY:

Implement subtyping for our language.

First, leave if-then-else untouched. Then, add joins and meets.