Type Systems

Lecture 5   Nov. 17th, 2004

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http://lampwww.epfl.ch/teaching/typeSystems/2004

Today:

1. Subtyping (functions and records)
2. Algorithmic Subtyping
3. Joins and Meets

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(λr:{x:Nat}. r.x) {x=0,y=1}
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ill-typed!

this function can ONLY be applied to records of the type {x:Nat}

→ the type system is WAY too strict! too much slack!

Actually, the function can be applied to

ANY record that has **at least the field x:Nat!**

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a **subtype** of {x:Nat}

Why "**sub**"?

{x:Nat,y:Nat} is a subtype, written **<:** of {x:Nat} because

# of records having this is LESS than # of records having this

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S T

→ If `t` satisfies S in some context, then it also satisfies T!!

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$\underset{S}{\{x:Nat,y:Nat\}} <: \underset{T}{\{x:Nat\}}$ (“is subtype of”)

→ If `t` satisfies S in some context, then it also satisfies T!!

is of type is of type

Rule of subsumtion:

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

… = “move to a more general type (supertype)”

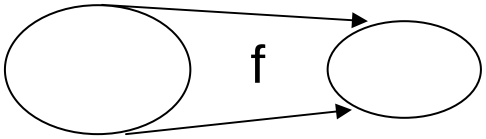
1. Subtyping (functions and records)

Rules for the Subtype Relation <:

reflexive, transitive: S <: S

$$\frac{S <: U \qquad U <: T}{S <: T}$$

“top type”: S <: Top



function f : $S_1 \rightarrow S_2$

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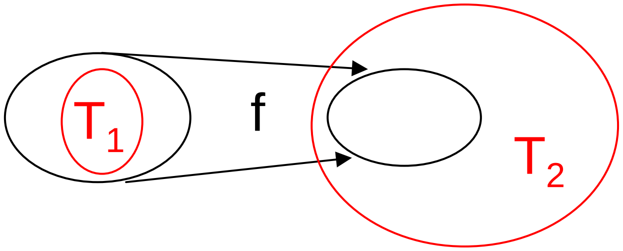

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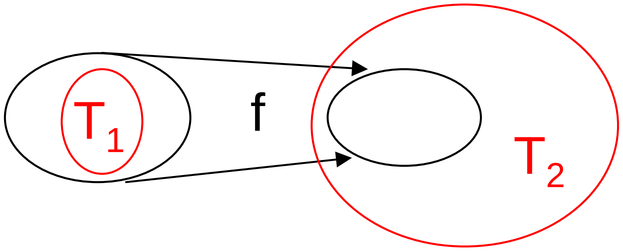

function $f : S_1 \rightarrow S_2$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \rightarrow S_2 <: T_1 \rightarrow T_2}$$

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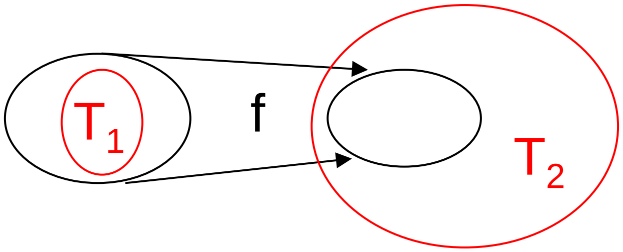

function f : $S_1 \to S_2$

contravariant
in argument →

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 <: T_1 \to T_2}$$

← covariant
in result

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$$S <: \mathrm{Top}$$

$$\frac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \rightarrow S_2 <: T_1 \rightarrow T_2}$$

records:

$$\{l_1{:}T_1, \ldots, l_{n+k}{:}T_{n+k}\} <: \{l_1{:}T_1, \ldots, l_n{:}T_n\}$$

(forget fields)

$$\frac{\text{for each } i \quad S_i <: T_i}{\{l_1{:}S_1, \ldots, l_n{:}S_n\} <: \{l_1{:}T_1, \ldots, l_n{:}T_n\}}$$

(subtype inside of record)

$$\frac{\{k_1{:}S_1, \ldots, k_n{:}S_n\} \text{ is permutation of } \{l_1{:}T_1, \ldots, l_n{:}T_n\}}{\{k_1{:}S_1, \ldots, k_n{:}S_n\} <: \{l_1{:}T_1, \ldots, l_n{:}T_n\}}$$

(don’t care
about order)

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`(λr:{x:Nat}. r.x) {x=0,y=1}`

ill-typed! (in simply lambda)

Can we type this now, using subtyping?

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$$\frac{\vdash \{x=0,y=1\}:\{x{:}Nat,y{:}Nat\} \qquad \{x{:}Nat,y{:}Nat\} <: \{x{:}Nat\}}{}$$

not derivable before
NOW: use *subsumption* !

$$\frac{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x):\{x{:}Nat\} \rightarrow Nat \qquad \vdash \{x=0,y=1\}:\{x{:}Nat\}}{\vdash (\lambda r{:}\{x{:}Nat\}.\ r.x)\ \{x=0,y=1\}}$$

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2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\frac{S <: U \qquad U <: T}{S <: T}$$

Are NOT syntax-directed!!

→ When to apply them??

$$S <: S$$

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

2. Algorithmic Subtyping

Problematic rules for implementing subtyping:

$$\frac{S <: U \qquad U <: T}{S <: T}$$

← needed to merge subtyping derivation of records

to see this, find a derivation for

$$S <: S$$

`{x:{a:Nat,b:Nat},y:{m:Nat}}`
`<:{x:{a:Nat}}`

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

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$$\cancel{\dfrac{S <: U \qquad U <: T}{S <: T}}$$

← needed to merge subtyping derivation of records

$$\cancel{S <: S}$$

$$\dfrac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}$$

new:

$$\dfrac{\begin{array}{c}\{k_1{:}S_1,\dots,k_n{:}S_n\} \subseteq \{l_1{:}T_1,\dots,l_m{:}T_m\} \\ k_j=l_i \text{ implies } S_j<:T_i\end{array}}{\{k_1{:}S_1,\dots,k_n{:}S_n\} <: \{l_1{:}T_1,\dots,l_m{:}T_m\}}$$

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ONLY needed for fu. application!

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$$\cancel{S <: S}$$

$$\cancel{\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}}$$

↑

ONLY needed for fu. application!

new:

$$\frac{\Gamma \vdash t_1{:}T_1 \to T_2 \qquad \Gamma \vdash t_2 : R \qquad R <: T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

2. Algorithmic Subtyping

Implementing subtyping:

$$\cancel{\dfrac{S <: U \qquad U <: T}{S <: T}}$$

$$\cancel{S <: S}$$

$$\cancel{\dfrac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T}}$$

$$\dfrac{T_1 <: S_1 \qquad S_2 <: T_2}{S_1 \to S_2 \;<:\; T_1 \to T_2}$$

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$$\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : R \qquad R <: T_1}{\Gamma \vdash t_1\ t_2 : T_2}$$

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AND: $S <: S$ for every base type S.

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→ are these rules *sound* and *complete* w.r.t. the previous rules ($\vdash$)??

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YES, prove this by straightforward induction!

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NO, $\vdash$ `{a=0}:{a:Nat,b:Nat}`,

but **not** true for $\Vdash$

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YES, prove this by straightforward induction!

Completeness: If $\Gamma \vdash t : T$, then $\Gamma \Vdash t : S$ for some $S <: T$

= "*Minimal Typing Theorem*"

→ Can you prove that S is actually **minimal**??

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
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What is the type of this term?

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or `{x:Top}`?

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take the *least* (most precise) common supertype of S and T

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or `Top`?

take the *least* (most precise) common supertype of S and T

= “the *join* of S and T”

=: $S \vee T$

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$$\frac{t_1 : \mathrm{Bool} \qquad t_2 : T_2 \qquad t_3 : T_3 \qquad T = T_2 \vee T_3}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T}$$

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$S \vee T$:= $\mathtt{Bool}$, if $S = T = \mathtt{Bool}$

$\{j_1{:}J_1, \ldots, j_q{:}J_q\}$ if $S = \{k_1{:}S_1, \ldots, k_m{:}S_m\}$,
$T = \{l_1{:}T_1, \ldots, l_n{:}T_n\}$,
$\{j_1{:}J_1, \ldots, j_q{:}J_q\} = S \cap T$, and
$J_u = S_v \vee T_w$ for $j_u = k_v = l_w$

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$A \wedge C$:=
meet of A and C
= greatest common
subtype

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$E \rightarrow F$, if $S = A \rightarrow B$ and $T = C \rightarrow D$ and
$E = A \wedge C$ and $F = B \vee D$

3. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

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$A \wedge C$:=
meet of A and C
= greatest common subtype

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$\{j_1{:}J_1, \ldots, j_q{:}J_q\}$ if $S = \{k_1{:}S_1, \ldots, k_m{:}S_m\}$,
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$E \rightarrow F$, if $S = A \rightarrow B$ and $T = C \rightarrow D$ and
$E = A \wedge C$ and $F = B \vee D$

Top, otherwise.

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$A \wedge C$:=
meet of A and C
= greatest common subtype

$S \wedge T$:=

define this in a similar way
as the join!

IN LAB TODAY:

Implement subtyping for our language.

First, leave if-then-else untouched. Then, add joins and meets.