

Type Systems

Lecture 2 Oct. 27th, 2004
Sebastian Maneth

<http://lampwww.epfl.ch/teaching/typeSystems/2004>

Today

1. What is the Lambda Calculus?
2. Its Syntax and Semantics
3. Church Booleans and Church Numerals
4. Lazy vs. Eager Evaluation (call-by-name vs. call-by-value)
5. Recursion
6. Nameless Implementation: deBruijn Indices

1. What is the Lambda Calculus

introduced in late 1930's by Alonzo Church and Stephen Kleene



used in 1936 by Church to prove the undecidability of the Entscheidungsproblem

is a formal system designed to investigate

- function definition
- function application
- recursion

1. What is the Lambda Calculus

introduced in late 1930's by Alonzo Church and Stephen Kleene

can compute the same as Turing Machines, which is everything we can (intuitively) compute (Church-Turing Thesis).

is a formal system designed to investigate

- function definition
- function application
- recursion

1. What is the Lambda Calculus

what do we want?

→ a **small core language**, into which other language constructs can be translated.

There are many such languages:

Turing Machines
μ-Recursive Functions
Chomsky's Type-0 Grammars
Cellular Automata
etc.

→ why do we pick out the Lambda Calculus?

because types are about values of **program variables**.

2. Syntax of the Lambda Calculus

Let V be a countable set of **variable names**.

The **set of lambda terms** (over V) is the smallest set T such that

1. if $x \in V$, then $x \in T$ variable
2. if $x \in V$ and $t_1 \in T$, then $\lambda x. t_1 \in T$ abstraction
3. if $t_1, t_2 \in T$, then $t_1 t_2 \in T$ application

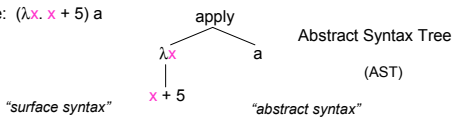
Function abstraction: instead of $f(x) = x + 5$
write $f = \lambda x. x + 5$

a **lambda term** (i.e., $\in T$)
representing a **nameless function**, which adds 5 to its parameter

2. Syntax of the Lambda Calculus

Function application: instead of $f(x)$
write $f\ x$

Example: $(\lambda x. x + 5)\ a$



Conventions (to save parenthesis)

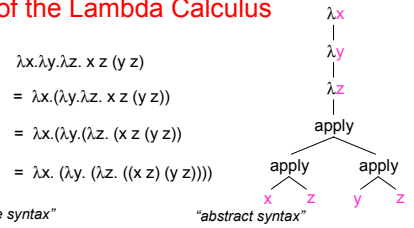
application is left associative: $x\ y\ z = (x\ y)\ z \neq x\ (y\ z)$

scope of **abstraction** extends **as far to the right as possible**:

$$\lambda x. x\ y = \lambda x. (x\ y) \neq (\lambda x. x)\ y$$

2. Syntax of the Lambda Calculus

Example:



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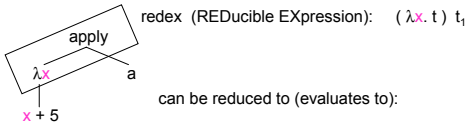
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2. Semantics of the Lambda Calculus

$(\lambda x. x + 5)\ a$ **SUBSTITUTE** a for x in $x + 5$



To compute in Lambda Calculus, ALL you do is **SUBSTITUTE!!**

2. Semantics of the Lambda Calculus

Example:

$$(\lambda x. \lambda y. f(y\ x))\ 5\ (\lambda x. x)$$

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$$\begin{aligned} & (\lambda x. \lambda y. f(y\ x))\ 5\ (\lambda x. x) \\ & = ((\lambda x. \lambda y. f(y\ x))\ 5)\ (\lambda x. x) \quad \text{because App binds to the left!} \end{aligned}$$

2. Semantics of the Lambda Calculus

Example:

$$\begin{aligned} & (\lambda x. \lambda y. f(y\ x))\ 5\ (\lambda x. x) \\ & = ((\lambda x. \lambda y. f(y\ x))\ 5)\ (\lambda x. x) \quad \text{because App binds to the left!} \\ & \xrightarrow{\beta\text{-red.}} [x \rightarrow 5](\lambda y. f(y\ x))\ (\lambda x. x) \\ & = (\lambda y. f(y\ 5))\ (\lambda x. x) \end{aligned}$$

2. Semantics of the Lambda Calculus

Example:

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 \xrightarrow{\beta\text{-red.}} & [x \rightarrow 5](\lambda y. f(y\ x))\ (\lambda x. x) \\
 = & (\lambda y. f(y\ 5))\ (\lambda x. x) \\
 \xrightarrow{\beta\text{-red.}} & [y \rightarrow \lambda x. x](f(y\ 5)) \\
 = & f(\lambda x. x\ 5)
 \end{aligned}$$

2. Semantics of the Lambda Calculus

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 \xrightarrow{\beta\text{-red.}} & [y \rightarrow \lambda x. x](f(y\ 5)) \\
 = & f(\lambda x. x\ 5) \\
 \xrightarrow{\beta\text{-red.}} & f\ 5 \quad (\text{normal form} = \text{cannot be reduced further})
 \end{aligned}$$

2. Semantics of the Lambda Calculus

Example: Does every λ -term have a normal form?

→ NO!!!

$$\begin{aligned}
 & (\lambda x. x\ x)\ (\lambda x. x\ x) \\
 \xrightarrow{\beta\text{-red.}} & [x \rightarrow (\lambda x. x\ x)]\ (x\ x)
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 \xrightarrow{\beta\text{-red.}} & (\lambda x. x\ x)\ (\lambda x. x\ x) \\
 & \vdots
 \end{aligned}$$

2. Semantics of the Lambda Calculus

Example: Does every λ -term have a normal form?

→ NO!!!

$$\underbrace{(\lambda x. x\ x)\ (\lambda x. x\ x)}_{\text{=: omega}} \quad \text{is called the } \omega \text{ combinator}$$

combinator = closed lambda term

= lambda term with *no free variables*

The simplest combinator, identity: $id := \lambda x. x$

2. Semantics of the Lambda Calculus

Free vs. Bound Variables:

$\lambda x. x y = \lambda x. \overset{\text{bound}}{\underbrace{(x y)}}^{\text{free}}$
 scope of x
 x is bound in its scope

Define the **set of free variables** of a term t, $FV(t)$, as

if $t = x \in V$, then $FV(t) = \{x\}$

if $t = \lambda x. t_1$, then $FV(t) = FV(t_1) \setminus \{x\}$

if $t = t_1 t_2$, then $FV(t) = FV(t_1) \cup FV(t_2)$

3. Church Booleans and Numerals

How to encode BOOLEANS into the lambda calculus?

tru → takes two arguments, selects the FIRST
 fls → takes two arguments, selects the SECOND

THEN: if-then-else can be defined as:

$\text{test } x \ u \ w = \text{"apply } x \text{ to } u \ w"$
 $= (\lambda k. \lambda m. \lambda n. k \ m \ n) \ x \ u \ w$
 $=: \text{test}$

$\text{tru} := \lambda m. \lambda n. m$

$\text{fls} := \lambda m. \lambda n. n$

$\text{test } \text{tru} \ u \ w \xrightarrow{\beta\text{-red.}} \dots \xrightarrow{\beta\text{-red.}} u$

3. Church Booleans and Numerals

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tru → takes two arguments, selects the FIRST
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$\text{tru} := \lambda m. \lambda n. m$

$\text{fls} := \lambda m. \lambda n. n$

$\text{test} := \lambda k. \lambda m. \lambda n. k \ m \ n$

How to do "and" on these BOOLEANS?

$\text{and } u \ w = \text{"apply } u \text{ to } w \ \text{fls}"$
 $:= (\lambda m. \lambda n. m \ n \ \text{fls}) \ u \ w$
 $=: \text{and}$

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 $:= (\lambda m. \lambda n. m \ n \ \text{fls}) \ u \ w$
 $=: \text{and}$

→ Define the **or** and **not** functions!

3. Church Booleans and Numerals

How to encode NUMBERS into the lambda calculus?

$c0 := \lambda s. \lambda z. z$

$c1 := \lambda s. \lambda z. s \ z$

$c2 := \lambda s. \lambda z. s \ (s \ z)$

$c3 := \lambda s. \lambda z. s \ (s \ (s \ z))$

etc.

THEN, the successor function can be defined as

$\text{scc} := \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$

$\text{scc } c0 \xrightarrow{\beta\text{-red.}} \lambda s. \lambda z. s \ (c0 \ s \ z) \xrightarrow{\beta\text{-red.}} \lambda s. \lambda z. s \ z = c1$

just like **fls**!
 Select the second argument.

3. Church Booleans and Numerals

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$c1 := \lambda s. \lambda z. s \ z$

$c2 := \lambda s. \lambda z. s \ (s \ z)$

$c3 := \lambda s. \lambda z. s \ (s \ (s \ z))$

$\text{scc} := \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)$

How to do "plus" and "times" on these Church Numerals?

$\text{plus} := \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)$

↑
"apply m times the successor to n"

3. Church Booleans and Numerals

How to encode NUMBERS into the lambda calculus?

```

c0 := λs. λz. z
c1 := λs. λz. s z       succ := λn. λs. λz. s (n s z)
c2 := λs. λz. s (s z)
c3 := λs. λz. s (s (s z))
  
```

How to do "plus" and "times" on these Church Numerals?

```

plus := λm. λn. λs. λz. m s (n s z)
      ↑
      "apply m times the successor to n"

times := λm. λn. m (plus n) c0
      ↑
      "apply m times (plus n) to c0"
  
```

3. Church Booleans and Numerals

How to encode NUMBERS into the lambda calculus?

```

c0 := λs. λz. z           succ := λn. λs. λz. s (n s z)
c1 := λs. λz. s z       plus := λm. λn. λs. λz. m s (n s z)
c2 := λs. λz. s (s z)
c3 := λs. λz. s (s (s z))
  
```

Questions:

- Write a function **subt** for subtraction on Church Numerals.
- How can other datatypes be encoded into the lambda calculus, like, e.g., **lists**, **trees**, **arrays**, and **variant records**?

4. Lazy vs. Eager Evaluation

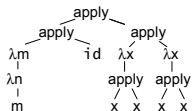
What does this lambda term evaluate to??

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tru id omega
```

4. Lazy vs. Eager Evaluation

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tru id omega
(λm. λn. m) (λx. x) ((λx. x x) (λx. x x))
→ where to start evaluating? which redex??
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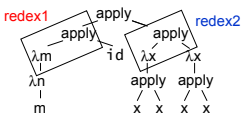


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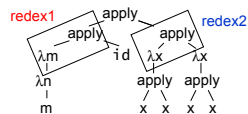


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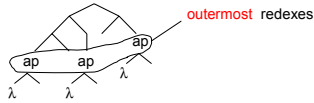
→ where to start evaluating? which redex??



→ if we always reduce redex2 then this lambda term has NO semantics.

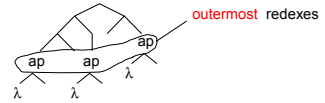
4. Lazy vs. Eager Evaluation

A redex if **outermost**, if in the AST it has no ancestor that is a redex.



4. Lazy vs. Eager Evaluation

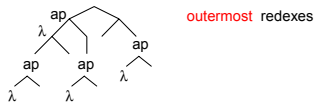
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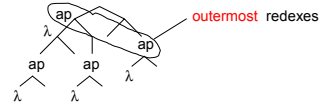
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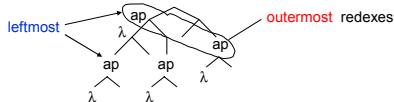
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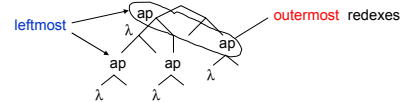
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4. Lazy vs. Eager Evaluation



Evaluation Strategies:

- normal order** always reduce **leftmost outermost** redex first
 - call-by-name** like normal order, but NOT inside abstractions
 - call-by-need** like call-by-name but with sharing
 - call-by-value** reduce only "value-redexes" (= argument is a value) and do this **leftmost**
- right branch of ap

4. Lazy vs. Eager Evaluation

Lazy seems better than **eager**, because more terms can be evaluated!

- can you define an infinite list consisting of all prime numbers?
(with **lazy** evaluation you can fetch the first n numbers of this list!)

```
> fetch c3 primelist
should compute the list [ 2, 3, 5 ]
```

If a term evaluates to a **normal form n** using **eager** evaluation, then it also evaluates to **n** using **lazy** evaluation.

- can you prove this!?
- what about the number of eval. steps needed by **eager** vs. **lazy**?

Lazy is hard to implement efficiently because copies of unevaluated lambda terms must be shared in order not to have duplicate reductions

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- can you prove this!?
- what about the number of eval. steps needed by **eager** vs. **lazy**?

Lazy is hard to implement it efficiently because lots of duplicate reductions might be done.

→ Most FL's use call-by-value. Also the TaPL book!

5. Recursion

```
fct = λn.if eq n c0 then c1 else (times n (fct (prd n)))
                                     ↑
                                     recursion
```

e.g. **fct c3** needs to **unroll** 4 times the definition
(expand)

```
fct c3 = if eq c3 c0 then c1 else (times c3 (
  if eq c2 c0 then c1 else (times c2 (
    if eq c1 c0 then c1 else (times c1 (
      if eq c0 c0 then c1 else (...))
    )
  )
)
( evaluates to c6 )
```

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```

→ Is there a **combinator** doing the **unrolling**, when applied to **fct**?

5. Recursion

```
fct = λn.if eq n c0 then c1 else (times n (fct (prd n)))
                                     ↑
                                     recursion
```

such a **combinator** is similar to **omega!** $(\lambda x. x x) (\lambda x. x x)$

- additionally to copying itself it should each time split of one application of the definition of **fct**

→ Is there a **combinator** doing the **unrolling**, when applied to **fct**?

5. Recursion

First, under call-by-name (**lazy**) evaluation.

(cbn) **fixed-point combinator** $Y := \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$

$g := \lambda fct. \lambda n. \text{if eq } n \text{ c0 then c1 else (times n (fct (prd n)))$

$Y g \text{ c3} \rightarrow$

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$$\begin{aligned} Y g \text{ c3} &\rightarrow (\lambda x. g (x x)) \underbrace{(\lambda x. g (x x))}_{=: h} \text{ c3} \\ &\rightarrow g (h h) \text{ c3} \end{aligned}$$

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lazy! $\rightarrow \lambda n. \text{if eq } n \text{ c0 then c1 else (times } n \text{ (hh (prd } n\text{))) } \text{ c3}$

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$$\begin{aligned} Y g \text{ c3} &\rightarrow (\lambda x. g (x x)) \underbrace{(\lambda x. g (x x))}_{=: h} \text{ c3} \\ &\rightarrow g (h h) \text{ c3 } \mathbf{eager!} \rightarrow g (g (h h)) \text{ c3} \rightarrow g(g(g(h h) c3) \dots \end{aligned}$$

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$\rightarrow \text{times } \text{c3} \text{ (hh (prd } \text{c3}\text{))}$

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$$Y g \text{ c3} \rightarrow (\lambda x. g (x x)) (\lambda x. g (x x)) \text{ c3}$$

$\underbrace{\hspace{10em}}_{=: h}$

$$\rightarrow g (h h) \text{ c3}$$

lazy! $\rightarrow \lambda n. \text{if eq } n \text{ c0 then c1 else (times } n \text{ (hh (prd } n\text{))) } \text{ c3}$

$\rightarrow \text{if eq } \text{c3} \text{ c0 then c1 else (times } \text{c3} \text{ (hh (prd } \text{c3}\text{)))}$

$\rightarrow \text{times } \text{c3} \text{ (hh (prd } \text{c3}\text{))}$

$\rightarrow \text{times } \text{c3} \text{ (g (h h) (prd } \text{c3}\text{))} \rightarrow \dots \rightarrow \text{times } \text{c3} \text{ c2 c1 c1}$

5. Recursion

Now, under **eager** (call-by-value) evaluation.

(cbv) **fixed-point combinator** $\text{fix} := \lambda f. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y))$

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5. Recursion

Question: Can you feel why the lambda calculus is Turing complete?

Can you prove it? What does it take to be Turing complete?

6. Nameless Implementation: deBruijn Indices

redex (REDucible EXpression): $(\lambda x. t) s$

β -reduction: $(\lambda x. t) s := [x \rightarrow s] t$

substitution **A.** only replace the FREE occurrences of x in t !
 $[x \rightarrow s] :$ **B.** if replacing within $(\lambda y. u)$ then y should NOT be FREE in s !

DEFINE $[x \rightarrow s] t$, by **induction** on the structure of t :

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2. $[x \rightarrow s] \lambda y. t_1 =$
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- $[x \rightarrow s] \lambda y. t_1 = \lambda y. [x \rightarrow s] t_1$ if $y \neq x$ and $y \notin \text{FV}(s)$ **A,B**
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→ to apply 2., **renaming** of BOUND y 's in t_1 might be necessary!!!
 = "alpha-conversion"

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Idea: let variable occurrences directly point to their binders, rather than referring to them by name.

→ use **natural numbers k** , meaning "the k -th enclosing λ "

e.g. $\lambda x. \lambda y. x (y x)$ BECOMES $\lambda. \lambda. 1 (0 1)$

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→ what to do with free variables??

use **naming context** $\Gamma \in V^*$. E.g., **bca** means **b**↔2, **c**↔1, **a**↔0

6. Nameless Implementation: deBruijn Indices

fix a **naming context** $\Gamma \in V^*$.

lambda term $\xleftrightarrow{\text{removenames}_\Gamma}$ nameless lambda term
 $\xleftrightarrow{\text{restorenames}_\Gamma}$
 $(\Gamma = xu) \lambda y. u y \xleftrightarrow{\text{remov}_\Gamma} \lambda. 1 0 \xleftrightarrow{\text{resto}_\Gamma} (\Gamma = xuy)$

substitution $[1 \rightarrow s](\lambda. 2)$ → increment all free vars in s by one!
 $\Gamma = xu \quad \uparrow \Gamma = \Gamma y$

$[j \rightarrow s](\lambda. t_1) = \lambda. [j+1 \rightarrow \text{shift}(1, s)] t_1$

shift function must keep track of BOUND vars in order to ONLY shift the FREE vars.

6. Nameless Implementation: deBruijn Indices

$\text{shift}(d, s) := \text{shiftb}(d, 0, s)$
 DON'T shift vars with index <0 !!

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$\text{shift}(d, s) := \text{shiftb}(d, 0, s)$
 DON'T shift vars with index <0 !!

$\text{shiftb}(d, b, k) = k$ if $k < b$, and $k+d$ otherwise
 $\text{shiftb}(d, b, \lambda. t_1) = \lambda. \text{shiftb}(d, b+1, t_1)$
 $\text{shiftb}(d, b, t_1 t_2) = \text{shiftb}(d, b, t_1) \text{shiftb}(d, b, t_2)$

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6. Nameless Implementation: deBruijn Indices

fix a **naming context** $\Gamma \in V^*$.

$\text{removenames}(\Gamma, x) =$ index of rightmost x in Γ
 $\text{removenames}(\Gamma, \lambda x. t_1) = \lambda. \text{removenames}(\Gamma x, t_1)$
 $\text{removenames}(\Gamma, t_1 t_2) = \text{removenames}(\Gamma, t_1) \text{removenames}(\Gamma, t_2)$

$\text{restorenames}(\Gamma, k) =$ k-th name in Γ
 $\text{restorenames}(\Gamma, \lambda. t) = \lambda x. \text{restorenames}(\Gamma x, t)$
 x is the first name not in Γ
 $\text{restorenames}(\Gamma, t_1 t_2) = \text{restorenames}(\Gamma, t_1) \text{restorenames}(\Gamma, t_2)$