Type Systems

Lecture 10 Dec. 22nd, 2004
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http://lampwww.epfl.ch/teaching/typeSystems/2004

Today F-sub: kernel / full

1. System F-sub
2. Properties of F-sub
3. Algorithmic Typing for F-Sub
4. Algorithmic *Sub*typing for F-Sub
5. Joins and Meets

1. System F-Sub

Bounded Quantification

```
f2poly = λX<:{a:Nat}. λx:X. {orig=x, asucc=succ(x.a)};
```

Has type `∀X<:{a:Nat}. X → {orig:X, asucc:Nat}`

1. System F-Sub

Bounded Quantification

```
f2poly = λX<:{a:Nat}. λx:X. {orig=x, asucc=succ(x.a)};
```

Has type `∀X<:{a:Nat}. X → {orig:X, asucc:Nat}`

How to derive it?

$$\vdash \lambda X<:\{a:Nat\}.\lambda x:X.\{orig=x,\ asucc=succ(x.a)\}:\ \forall X<:\{a:Nat\}.\ X \to \{orig:X,\ asucc:Nat\}$$

1. System F-Sub

Bounded Quantification

type abstraction (remove ∀)

If bound satisfied, then term has specified type

$$\frac{X<:\{a:Nat\} \vdash \lambda x:X.\{orig=x,\ asucc=succ(x.a)\} : X \to \{orig:X,\ asucc:Nat\}}{\vdash \lambda X<:\{a:Nat\}.\lambda x:X.\{orig=x,\ asucc=succ(x.a)\}:\ \forall X<:\{a:Nat\}.\ X \to \{orig:X,\ asucc:Nat\}}$$

1. System F-Sub

lambda abstraction (remove →)

If argument type satisfied, then result term has specified result type.

$$\frac{\frac{X<:\{a:Nat\},x:X \vdash \{orig=x,\ asucc=succ(x.a)\} : \{orig:X,\ asucc:Nat\}}{X<:\{a:Nat\} \vdash \lambda x:X.\{orig=x,\ asucc=succ(x.a)\}:X \to \{orig:X,\ asucc:Nat\}}}{\vdash \lambda X<:\{a:Nat\}.\lambda x:X.\{orig=x,\ asucc=succ(x.a)\}:\ \forall X<:\{a:Nat\}.\ X \to \{orig:X,\ asucc:Nat\}}$$

1. System F-Sub

make record (remove { })

field terms have specified types

$X<:\{a:Nat\},x:X \vdash x:X$ $\quad$ $X<:\{a:Nat\},x:X \vdash succ(x.a):Nat$

$X<:\{a:Nat\},x:X \vdash \{orig=x, asucc=succ(x.a)\} : \{orig:X, asucc:Nat\}$

$X<:\{a:Nat\} \vdash \lambda x:X.\{orig=x, asucc=succ(x.a)\}:X \to \{orig:X, asucc:Nat\}$

$\vdash \lambda X<:\{a:Nat\}.\lambda x:X.\{orig=x, asucc=succ(x.a)\}: \forall X<:\{a:Nat\}.\ X \to \{orig:X, asucc:Nat\}$

1. System F-Sub

what now?

ok

$X<:\{a:Nat\},x:X \vdash x:X$ $\quad$ $X<:\{a:Nat\},x:X \vdash x:\{a:Nat\}$ $\quad$ $X<:\{a:Nat\},x:X \vdash x.a:Nat$ $\quad$ $X<:\{a:Nat\},x:X \vdash succ(x.a):Nat$

$X<:\{a:Nat\},x:X \vdash \{orig=x, asucc=succ(x.a)\} : \{orig:X, asucc:Nat\}$

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1. System F-Sub

what now? → subsumption!

term may be of any subtype

ok

$X<:\{a:Nat\},x:X \vdash x:X$ $\quad$ $X<:\{a:Nat\},x:X \vdash X<:\{a:Nat\}$ $\quad$ $X<:\{a:Nat\},x:X \vdash x:\{a:Nat\}$ $\quad$ $X<:\{a:Nat\},x:X \vdash x.a:Nat$ $\quad$ $X<:\{a:Nat\},x:X \vdash succ(x.a):Nat$

$X<:\{a:Nat\},x:X \vdash \{orig=x, asucc=succ(x.a)\} : \{orig:X, asucc:Nat\}$

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1. System F-Sub

ok ok

$X<:\{a:Nat\},x:X \vdash x:X$ $\quad$ $X<:\{a:Nat\},x:X \vdash X<:\{a:Nat\}$ $\quad$ $X<:\{a:Nat\},x:X \vdash x:\{a:Nat\}$ $\quad$ $X<:\{a:Nat\},x:X \vdash x.a:Nat$ $\quad$ $X<:\{a:Nat\},x:X \vdash succ(x.a):Nat$

$X<:\{a:Nat\},x:X \vdash \{orig=x, asucc=succ(x.a)\} : \{orig:X, asucc:Nat\}$

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1. System F-Sub

new typing rule

type abstraction (remove ∀)

If bound satisfied, then term has specified type

$$\frac{\Gamma, X<:B \vdash t:T}{\Gamma \vdash \lambda X<:B.t : \forall X<:B.T}$$

1. System F-Sub

As in System F: to **apply** a polymorphic function, we have to supply a concrete SUBtype C.

```
( f2poly [{a:Nat, b:Bool}] ) {a=5, b=true}
```

1. System F-Sub

As in System F: to **apply** a polymorphic function, we have to supply a concrete SUBtype C.

C<:{a:Nat}

∀X<:{a:Nat}. X → {orig:X, asucc:Nat}

f2poly [{a:Nat, b:Bool}]

{a:Nat, b:Bool} → {orig:{a:Nat,b:Bool}, asucc:Nat}

1. System F-Sub

How to derive it?

⊢ f2poly [{a:Nat, b:Bool}]:
{a:Nat, b:Bool} → {orig:{a:Nat,b:Bool}, asucc:Nat}

1. System F-Sub

How to derive it?

Applying [X/{a:Nat, b:Bool}] must give specified type

⊢ f2poly: ∀X<:{a:Nat}.X→{orig:X, asucc:Nat} ⊢ {a:Nat, b:Bool} <:{a:Nat}

⊢ f2poly [{a:Nat, b:Bool}]:
{a:Nat, b:Bool} → {orig:{a:Nat,b:Bool}, asucc:Nat}

1. System F-Sub

(record subtyping)

ok ok

⊢ f2poly: ∀X<:{a:Nat}.X→{orig:X, asucc:Nat} ⊢ {a:Nat, b:Bool} <:{a:Nat}

⊢ f2poly [{a:Nat, b:Bool}]:
{a:Nat, b:Bool} → {orig:{a:Nat,b:Bool}, asucc:Nat}

1. System F-Sub

new typing rule

type application (remove [ ])

t polymorphic with bound X<:B C subtype of B

$$\frac{\Gamma \vdash t : \forall X<:B.T \quad \Gamma \vdash C<:B}{\Gamma \vdash t[C] : [X/C]T}$$

1. System F-Sub

Typing Rules

- → Usual lambda-rules (T-VAR, T-ABS, T-APP)
- → type abstraction
- → type application (uses subtyping on bounds!)
- → subsumption (uses subtyping!)

1. System F-Sub

Subtyping Rules

- → reflexivity, transitivity, Top (evth. is <:Top)
- → type variables: $\dfrac{X<:T \in \Gamma}{\Gamma \vdash X<:T}$
- → function (S-ARROW)

  (covariant on result,
  contravariant on argument)

1. System F-Sub

new subtyping rule (for quantified types)

$$\Gamma \vdash \forall X<:B_1.T_1 \;<:\; \forall X<:B_2.T_2$$

1. System F-Sub

new subtyping rule (for quantified types)

Which bound??

$$\frac{\Gamma, X<:B \vdash T_1 <: T_2}{\Gamma \vdash \forall X<:B_1.T_1 \;<:\; \forall X<:B_2.T_2}$$

1. System F-Sub

new subtyping rule (for quantified types)

Must have SAME bound B

"the *kernel* rule"
$$\frac{\Gamma, X<:B \vdash T_1 <: T_2}{\Gamma \vdash \forall X<:B.T_1 \;<:\; \forall X<:B.T_2}$$

→ With this simple rule: **Kernel F-Sub**

1. System F-Sub

"the *kernel* rule"
$$\frac{\Gamma, X<:B \vdash T_1 <: T_2}{\Gamma \vdash \forall X<:B.T_1 \;<:\; \forall X<:B.T_2}$$

∀X<:{a:Nat}. {a:Nat,b:Bool} <: ∀X<:{a:Nat}. {a:Nat}

If expected type is ∀X<:{a:Nat}. {a:Nat}
then also fu. of type ∀X<:{a:Nat}. {a:Nat,b:Bool} is OK!

1. System F-Sub

"the *kernel* rule"
$$\frac{\Gamma, X<:B \vdash T_1 <: T_2}{\Gamma \vdash \forall X<:B.T_1 \;<:\; \forall X<:B.T_2}$$

∀X<:{a:Nat}. {a:Nat,b:Bool} <: ∀X<:{a:Nat}. {a:Nat}

If expected type is ∀X<:{a:Nat}. {a:Nat}
then also fu. of type ∀X<:{a:Nat}. {a:Nat,b:Bool} is OK!

→ Will only be instantiated by subtypes X of {a:Nat}

THUS, ∀X<:Top is OK too, because it asks for less! (for the least, actually..)

∀X<:{a:Nat,d:Nat} NOT OK, because we only know X will be <:{a:Nat}

1. System F-Sub

Full F-Sub

$$\frac{\Gamma \vdash B_2 <: B_1 \qquad \Gamma, X<:B_2 \vdash T_1 <: T_2}{\Gamma \vdash \forall X<:B_1.T_1 \;<:\; \forall X<:B_2.T_2}$$

→ covariant on result

→ contravariant on bounds

2. Properties of F-Sub

Preservation
If $t \vdash t:T$ and $t \to t'$, then $\Gamma \vdash t':T$.

Progress
If t is a closed and well-typed term, then either
t is a value, or
$t \to t'$ for some term t'.

Proofs: Induction on the structure of terms.

→ Use canonical forms lemma:

If v is closed value of type $T_1 \to T_2$, then $v = \lambda x:S_1.t_2$
If v is closed value of type $\forall X<:T_1.T_2$, then $v = \lambda X<:T_1.t_2$.

3. Algorithmic Typing for F-Sub

Idea of type checking algorithm for simply typed lambda-calculus w. subtyping:

→ calculate the *minimal type* of terms

Apply same idea for F-Sub!

Now application, and then subsumtion, and all is OK!

$$\frac{\dfrac{X<:Nat\to Nat,\ y:X \vdash y\ 5:\ Nat}{X<:Nat\to Nat \vdash \lambda y:X.\ y\ 5:\ X \to Nat}}{\vdash \lambda X<:Nat\to Nat.\ \lambda y:X.\ y\ 5\ :\ \forall X.X<:Nat\to Nat.\ X \to Nat}$$

3. Algorithmic Typing for F-Sub

Idea of type checking algorithm for simply typed lambda-calculus w. subtyping:

→ calculate the *minimal type* of terms

Apply same idea for F-Sub!

→ subsumtion is NOT syntax-directed!! Can we do without??

Now application, and then subsumtion, and all is OK!

$$\frac{\dfrac{X<:Nat\to Nat,\ y:X \vdash y\ 5:\ Nat}{X<:Nat\to Nat \vdash \lambda y:X.\ y\ 5:\ X \to Nat}}{\vdash \lambda X<:Nat\to Nat.\ \lambda y:X.\ y\ 5\ :\ \forall X.X<:Nat\to Nat.\ X \to Nat}$$

3. Algorithmic Typing for F-Sub

→ subsumtion is NOT syntax-directed!! Can we do without??

Now application, and then subsumtion, and all is OK!

$X<:Nat\to Nat,\ y:X \vdash y\ 5:\ Nat$

→ y must be arrow type!

→ smallest (non-variable) arrow type, that is a supertype of X

$\Gamma \vdash X \Uparrow Nat\to Nat$

"X exposes to Nat→Nat under Γ"

3. Algorithmic Typing for F-Sub

Exposure:

$$\frac{X<:T \in \Gamma \qquad \Gamma \vdash T \Uparrow T'}{\Gamma \vdash X \Uparrow T'} \qquad \frac{T \text{ is not a type variable}}{\Gamma \vdash T \Uparrow T}$$

Example: Γ = X<:Nat, Y<:Nat→Nat, Z<:Y, W<:Z

Then $\Gamma \vdash Z \Uparrow Nat\to Nat$ and $\Gamma \vdash W \Uparrow Nat\to Nat$

Lemma. If $\Gamma \vdash S \Uparrow T$, then
(1) $\Gamma \vdash S<:T$
(2) If $\Gamma \vdash S<:U$ and U is not a variable, then $\Gamma \vdash T<:U$.

3. Algorithmic Typing for F-Sub

New rule for application, includes argument subsumtion:

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash T_1 \Uparrow (D \to E) \qquad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash T_2 <: D}{\Gamma \vdash t_1\ t_2 : E}\ \text{TA-APP}$$

In Example:

$$\frac{\Gamma \vdash y : X \quad \Gamma \vdash X \Uparrow (Nat \to Nat) \qquad \Gamma \vdash 5 : Nat \quad \Gamma \vdash Nat <: Nat}{\underbrace{X <: Nat \to Nat,\ y : X}_{\Gamma} \vdash y\ 5 : Nat}\ \text{TA-APP}$$

3. Algorithmic Typing for F-Sub

New rule for type application, includes argument subsumtion:

Old:
$$\frac{\Gamma \vdash t : \forall X <: B.T \quad \Gamma \vdash C <: B}{\Gamma \vdash t\ [C] : [X/C]T}$$

New:
$$\frac{\Gamma \vdash t : T_1 \quad \Gamma \vdash T_1 \Uparrow \forall X <: B.T \qquad \Gamma \vdash C <: B}{\Gamma \vdash t\ [C] : [X/C]T}\ \text{TA-TAPP}$$

3. Algorithmic Typing for F-Sub

$\vdash$ = T-VAR, T-ABS, T-APP, T-TABS, T-TAPP, T-SUB

(T-VAR, T-ABS, T-APP: as in simply typed)

$\vdash$ = T-VAR, T-ABS, TA-APP, T-TABS, TA-TAPP

Theorem. (correctness of minimal typing)

(1) If $\Gamma \vdash t : T$, then $\Gamma \vdash t : T$ (soundness)

(2) If $\Gamma \vdash t : T$, then $\Gamma \vdash t : M$ with $\Gamma \vdash M <: T$. (completeness)

Corollary. The relation $\vdash$ is decidable, given a decision procedure for the subtype relation.

4. Algorithmic *Sub*typing for F-Sub

Subtyping Rules

- reflexivity, transitivity, Top (evth. is <:Top)
- type variables
- function (S-ARROW)
- quantified types (kernel / full)

4. Algorithmic *Sub*typing for F-Sub

Problematic Subtyping Rules

- reflexivity, transitivity, Top
- type variables
- function (S-ARROW)
- quantified types (kernel / full)

4. Algorithmic *Sub*typing for F-Sub

Problematic Subtyping Rules

- reflexivity, transitivity, Top
- type variables
- function (S-ARROW)
- quantified types (kernel / full)

Reflexivity is ONLY needed for variables: $\Gamma \vdash X <: X$ not derivable without it!

- Replace reflexivity by **new rule**: $\Gamma \vdash X <: X$.

4. Algorithmic *Sub*typing for F-Sub

Problematic Subtyping Rules

- → $\Gamma \vdash X<:X$, transitivity, Top
- → type variables
- → function (S-ARROW)
- → quantified types (kernel / full)

Transitivity: $\Gamma = W<:Top, V<:W, U<:V, X<:U$

$\Gamma \vdash Z<:W$ can ONLY be proved using transitivity.

$$\dfrac{\dfrac{X<:U \in \Gamma}{\Gamma \vdash X<:U} \qquad \vdots \; \Gamma \vdash U<:W}{\Gamma \vdash X<:W}$$

4. Algorithmic *Sub*typing for F-Sub

Problematic Subtyping Rules

- → $\Gamma \vdash X<:X$, transitivity, Top
- → type variables
- → function (S-ARROW)
- → quantified types (kernel / full)

Transitivity: $\Gamma = T<:Top, V<:T, U<:V, X<:U$

$\Gamma \vdash Z<:W$ can ONLY be proved using transitivity.

$$\dfrac{\dfrac{X<:U \in \Gamma}{\Gamma \vdash X<:U} \qquad \vdots \; \Gamma \vdash U<:T}{\Gamma \vdash X<:T}$$

← ONLY essential use of transitivity!

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules

- → $\Gamma \vdash X<:X$, Top
- → function (S-ARROW)
- → quantified types (kernel / full)

→ Replace transitivity, and type vars by **new rule**:

$$\dfrac{X<:U \in \Gamma \qquad \Gamma \vdash U<:T}{\Gamma \vdash X<:T}$$

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (kernel)

$$\dfrac{}{\Gamma \vdash S<:Top} \qquad \dfrac{X<:U \in \Gamma \qquad \Gamma \vdash U<:T}{\Gamma \vdash X<:T}$$

$$\dfrac{}{\Gamma \vdash X<:X}$$

$$\dfrac{\Gamma \vdash T_1<:S_1 \qquad \Gamma \vdash S_2<:T_2}{\Gamma \vdash S_1 \to S_2 <: T_1 \to T_2} \qquad \dfrac{\Gamma, X<:B \vdash T_1<:T_2}{\Gamma \vdash \forall X<:B.T_1 <: \forall X<:B.T_2}$$

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (kernel)

$$\dfrac{}{\Gamma \vdash S<:Top} \qquad \dfrac{X<:U \in \Gamma \qquad \Gamma \vdash U<:T}{\Gamma \vdash X<:T}$$

$$\dfrac{}{\Gamma \vdash X<:X}$$

$$\dfrac{\Gamma \vdash T_1<:S_1 \qquad \Gamma \vdash S_2<:T_2}{\Gamma \vdash S_1 \to S_2 <: T_1 \to T_2} \qquad \dfrac{\Gamma, X<:B \vdash T_1<:T_2}{\Gamma \vdash \forall X<:B.T_1 <: \forall X<:B.T_2}$$

Theorem. $\Gamma \vdash S<:T$ if and only if $\Gamma \vdash S<:T$.

Theorem. The subtyping algorithm terminates on all inputs.

→ Subtyping in kernel F-sub is decidable.

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (FULL)

$$\dfrac{}{\Gamma \vdash S<:Top} \qquad \dfrac{X<:U \in \Gamma \qquad \Gamma \vdash U<:T}{\Gamma \vdash X<:T}$$

$$\dfrac{}{\Gamma \vdash X<:X}$$

$$\dfrac{\Gamma \vdash T_1<:S_1 \qquad \Gamma \vdash S_2<:T_2}{\Gamma \vdash S_1 \to S_2 <: T_1 \to T_2} \qquad \cancel{\dfrac{\Gamma, X<:B \vdash T_1<:T_2}{\Gamma \vdash \forall X<:B.T_1 <: \forall X<:B.T_2}}$$

$$\dfrac{\Gamma \vdash B_2<:B_1 \qquad \Gamma, X<:B_2 \vdash T_1<:T_2}{\Gamma \vdash \forall X<:B_1.T_1 <: \forall X<:B_2.T_2}$$

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (FULL) **do not terminate!!**
→Construct a vicious circle:

Define $\neg S = \forall X<:S.X$

Then, $\Gamma \vdash \neg S <: \neg T$ iff $T<:S$. (Contravariance on bounds..)

$T := \forall X<:Top.\ \neg(\forall Y<:X.\ \neg Y)$

Try to show that

$X_0<:T \vdash X_0 <: \forall X_1<:X_0.\neg X_1$

4. Algorithmic *Sub*typing for F-Sub

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Try to show that

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$X_0<:T \vdash \forall X_1<:Top.\neg(\forall X_2<:X_1.\neg X_2) <: \forall X_1<:X_0.\neg X_1$
$X_0<:T, X_1<:X_0 \vdash \neg(\forall X_2<:X_1.\neg X_2) <: \neg X_1$

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (FULL) **do not terminate!!**
→Construct a vicious circle:

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Try to show that

$X_0<:T \vdash X_0 <: \forall X_1<:X_0.\neg X_1$
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$X_0<:T, X_1<:X_0 \vdash \neg(\forall X_2<:X_1.\neg X_2) <: \neg X_1$
$X_0<:T, X_1<:X_0 \vdash X_1 <: \forall X_2<:X_1.\neg X_2.$

4. Algorithmic *Sub*typing for F-Sub

Algorithmic Subtyping Rules (FULL) **do not terminate!!**
→Construct a vicious circle:

Define $\neg S = \forall X<:S.X$

Then, $\Gamma \vdash \neg S <: \neg T$ iff $T<:S$. (Contravariance on bounds..)

$T := \forall X<:Top.\ \neg(\forall Y<:X.\ \neg Y)$

Try to show that

$X_0<:T \vdash X_0 <: \forall X_1<:X_0.\neg X_1$
$X_0<:T \vdash \forall X_1<:Top.\neg(\forall X_2<:X_1.\neg X_2) <: \forall X_1<:X_0.\neg X_1$
$X_0<:T, X_1<:X_0 \vdash \neg(\forall X_2<:X_1.\neg X_2) <: \neg X_1$
$X_0<:T, X_1<:X_0 \vdash X_1 <: \forall X_2<:X_1.\neg X_2$

5. Joins and Meets

How to type if-then-else, in the presence of subsumtion?

```
if true then {x=true,y=false} else {x=false,z=true}
```

What is the type of this term?

→ {x:Bool}
<:
or {x:Top}?
<:
or {}?
<:
or Top?

take the *least* (most precise) common supertype of S and T

= "the *join* of S and T"

=: S ∨ T

$$\frac{t_1 : \text{Bool} \quad t_2 : T_2 \quad t_3 : T_3 \quad T = T_2 \vee T_3}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T}$$

5. Joins and Meets

$\Gamma \vdash S \vee T :=$

T	if $\Gamma \vdash S<:T$
S	if $\Gamma \vdash T<:S$
J	if $S=X$, $X<:U \in \Gamma$, and $\Gamma \vdash U \vee T=J$
J	if $T=X$, $X<:U \in \Gamma$, and $\Gamma \vdash S \vee U=J$
$M \to J$	if $S=S_1 \to S_2$, $T=T_1 \to T_2$, $\Gamma \vdash S_1 \wedge T_1=M$, and $\Gamma \vdash S_1 \vee T_2=J$
$\forall X<:U.J_2$	if $S= \forall X<:U.S_2$, $T= \forall X<:U.T_2$, $\Gamma, X<:U \vdash S_2 \vee T_2=J_2$
Top	otherwise

5. Joins and Meets

In kernel F-sub:

→ every pair S,T has (effectively) a join

→ if S and T have at least one subtype in common, then they have (effectively) a meet

In full F-sub: NO / NO

Summary

formalism	comput. power	type checking	type reconstr.
Simply typed lambda calculus	normalizing	lin.time	poly.time
Simply typed lambda calculus + subtyping	normalizing	lin.time	poly.time
Featherweight Java	r.e. compl.	lin.time	
Let-Polymorphism / Prenex-Polymorphism	normalizing	lin.time	EXPTIME
System F	normalizing	lin.time	UNDEC.
System F + subtyping (kernel)	normalizing	poly.time	UNDEC.
System F + subtyping (full)	normalizing	UNDEC.	UNDEC.