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n-ary Semaphores [Mil99: Ex. 5.14]

$$\begin{aligned} S^{(1)} &\stackrel{\text{def}}{=} p.S_1^{(1)} \\ S_1^{(1)} &\stackrel{\text{def}}{=} v.S^{(1)} \end{aligned}$$

$$\begin{aligned} S^{(2)} &\stackrel{\text{def}}{=} p.S_1^{(2)} \\ S_1^{(2)} &\stackrel{\text{def}}{=} p.S_2^{(2)} + v.S^{(2)} \\ S_2^{(2)} &\stackrel{\text{def}}{=} v.S_1^{(2)} \end{aligned}$$

Proposition: $S^{(1)} \mid S^{(1)} \sim S^{(2)}$.

Proof ?

$$\mathcal{R} \stackrel{\text{def}}{=} \{ \ (S^{(1)}|S^{(1)}, S^{(2)}), \ (S_1^{(1)}|S_0^{(1)}, S_1^{(2)}), \ (S_1^{(1)}|S_1^{(1)}, S_2^{(2)}), \ (S_0^{(1)}|S_1^{(1)}, S_1^{(2)}) \ \}$$

Bisimulation up to $\equiv$

Definition:

A binary relation $\mathcal{S}$ over $\mathcal{P}$ is a **strong simulation up to** $\equiv$ if, whenever $P \mathrel{\mathcal{S}} Q$:

if $p \stackrel{\mu}{\longrightarrow} p'$, then there is $q' \in Q$ such that $q \stackrel{\mu}{\longrightarrow} q'$ and $p' \equiv\mathcal{S}\equiv q'$.

Strong bisimulation up to $\equiv \ldots$

Proposition:

Let $\mathcal{S}$ be a strong simulation up to $\equiv$.
If $P \mathrel{\mathcal{S}} Q$, then $P \sim Q$.

Bisimulation up to $\equiv$

Check !

$$\mathcal{R}_1 \stackrel{\text{def}}{=} \{\ (S^{(1)}|S^{(1)}, S^{(2)}),\ (S_1^{(1)}|S_1^{(1)}, S_2^{(2)}),\ \ (S_0^{(1)}|S_1^{(1)}, S_1^{(2)})\ \}$$

and

$$\mathcal{R}_2 \stackrel{\text{def}}{=} \{\ (S^{(1)}|S^{(1)}, S^{(2)}),\ \ (S_1^{(1)}|S_0^{(1)}, S_1^{(2)}),\ (S_1^{(1)}|S_1^{(1)}, S_2^{(2)})\ \}$$

are both strong bisimulations up to $\equiv$.

Towards Observation Equivalence

Different internal behavior should not count !

Definition: (observations / weak actions)

1. $\Rightarrow \stackrel{\text{def}}{=} \longrightarrow^{*}$

2.
   - $\stackrel{\lambda}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow \stackrel{\lambda}{\longrightarrow} \Rightarrow$
   - $\stackrel{\tau}{\Longrightarrow} \stackrel{\text{def}}{=} \longrightarrow^{+} \quad = \quad \Rightarrow \stackrel{\tau}{\longrightarrow} \Rightarrow$

3. Let $e = \lambda_1 \cdots \lambda_n$. (If $n = 0$ then $e = \epsilon$.)
   - $\stackrel{\epsilon}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow$
   - $\stackrel{\lambda_1 \cdots \lambda_n}{\Longrightarrow} \stackrel{\text{def}}{=} \stackrel{\lambda_1}{\Longrightarrow} \cdots \stackrel{\lambda_n}{\Longrightarrow}$

Weak Simulation

Definition:

Let $\mathcal{S}$ be a binary relation over $\mathcal{P}$.
$\mathcal{S}$ is a weak simulation if, whenever $P \; \mathcal{S} \; Q$,
if $P \stackrel{e}{\Longrightarrow} P'$ then there is $Q' \in \mathcal{P}$
such that $Q \stackrel{e}{\Longrightarrow} Q'$ and $P' \; \mathcal{S} \; Q'$.

Q weakly simulates P
if there is a weak simulation $\mathcal{S}$ with $P \; \mathcal{S} \; Q$.

Example:

Prove that $Q = \tau.a.\tau.b.Q$ simulates $P = a.b.P$.

Weak Simulation (II)

Proposition:

$\mathcal{S}$ is a weak simulation **iff**, whenever $P \, \mathcal{S} \, Q$,

- if $P \longrightarrow P'$ then there is $Q' \in \mathcal{P}$ such that $Q \Rightarrow Q'$ and $P' \, \mathcal{S} \, Q'$.
- if $P \xrightarrow{\lambda} P'$ then there is $Q' \in \mathcal{P}$ such that $Q \stackrel{\lambda}{\Longrightarrow} Q'$ and $P' \, \mathcal{S} \, Q'$.

Weak Bisimulation

<u>**Definition:**</u>

$\ldots$ (* straightforward / no surprise *)

P and Q are **weakly bisimilar**,
weakly equivalent, or **observation equivalent**,
written $P \approx Q$,
if there exists a weak bisimulation $\mathcal{B}$ with $P \mathrel{\mathcal{B}} Q$.

<u>**Proposition:**</u>

1. $\approx$ is an equivalence relation.
2. $\approx$ is itself a weak bisimulation.

Strong vs Weak

1. every strong simulation is also a weak one
2. $P \sim Q$ implies $P \approx Q$
3. see examples later on . . .

Weak simulation up to $\sim$

Definition:
$\mathcal{S}$ is a weak simulation up to $\sim$
if, whenever $P \; \mathcal{S} \; Q$,

- if $P \longrightarrow P'$ then there is $Q' \in \mathcal{P}$
  such that $Q \Rightarrow Q'$ and $P' \sim\mathcal{S}\sim Q'$.
- if $P \xrightarrow{\lambda} P'$ then there is $Q' \in \mathcal{P}$
  such that $Q \stackrel{\lambda}{\Longrightarrow} Q'$ and $P' \sim\mathcal{S}\sim Q'$.

$\mathcal{S}$ is a weak bisimulation up to $\sim$
if its converse also has this property.

Weak simulation up to $\sim$ (II)

Proposition:

If $\mathcal{B}$ is a weak bisimulation up to $\sim$ and $P\ \mathcal{B}\ Q$, then $P \approx Q$.

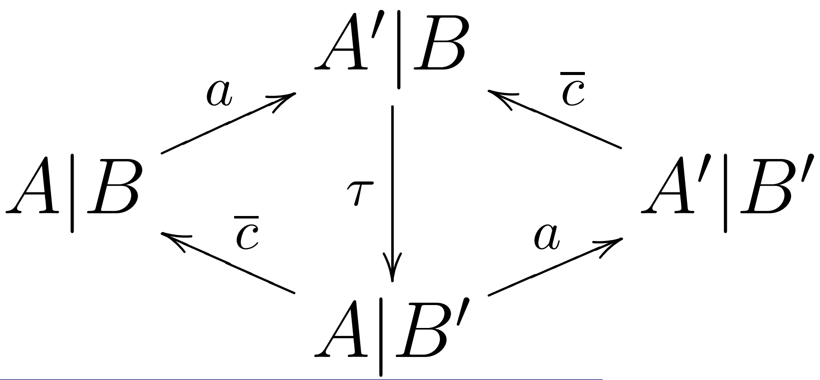
Proof ?

Example

$$
\begin{aligned}
A &\stackrel{\text{def}}{=} a.A' \quad (= a.\overline{b}.A) \\
A' &\stackrel{\text{def}}{=} \overline{b}.A \\
B &\stackrel{\text{def}}{=} b.B' \quad (= b.\overline{c}.B) \\
B' &\stackrel{\text{def}}{=} \overline{c}.B
\end{aligned}
$$

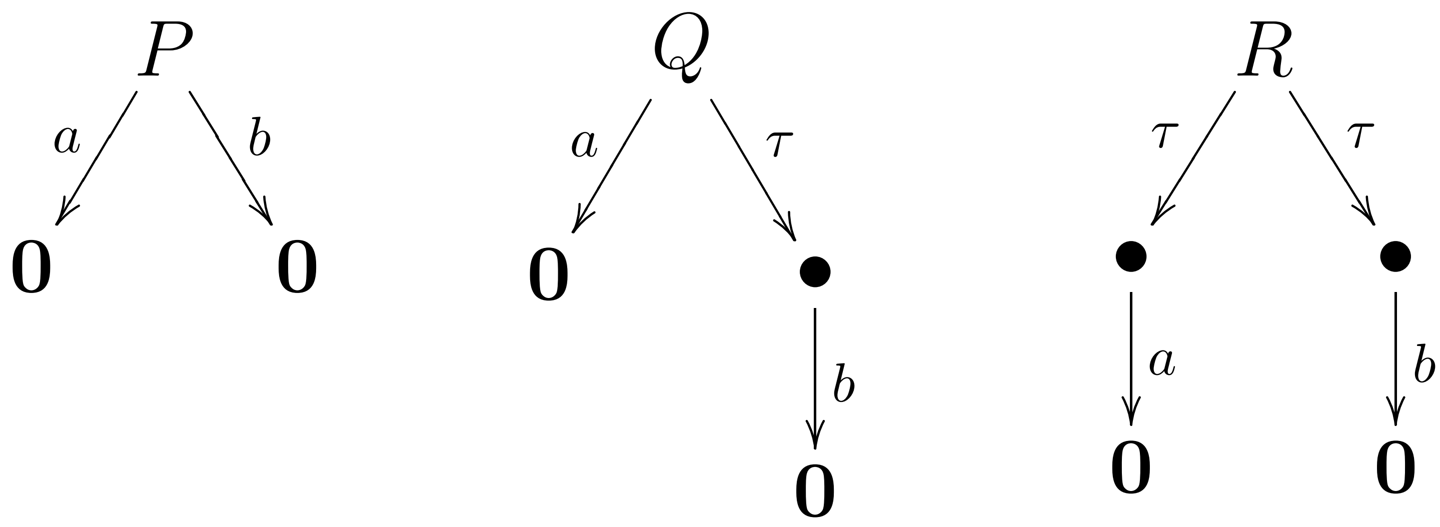
$$
\begin{aligned}
E &\stackrel{\text{def}}{=} a.E' \\
E' &\stackrel{\text{def}}{=} a.E'' + \overline{c}.E \\
E'' &\stackrel{\text{def}}{=} \overline{c}.E'
\end{aligned}
$$

Prove that $(\boldsymbol{\nu} b)(A|B) \approx E$.



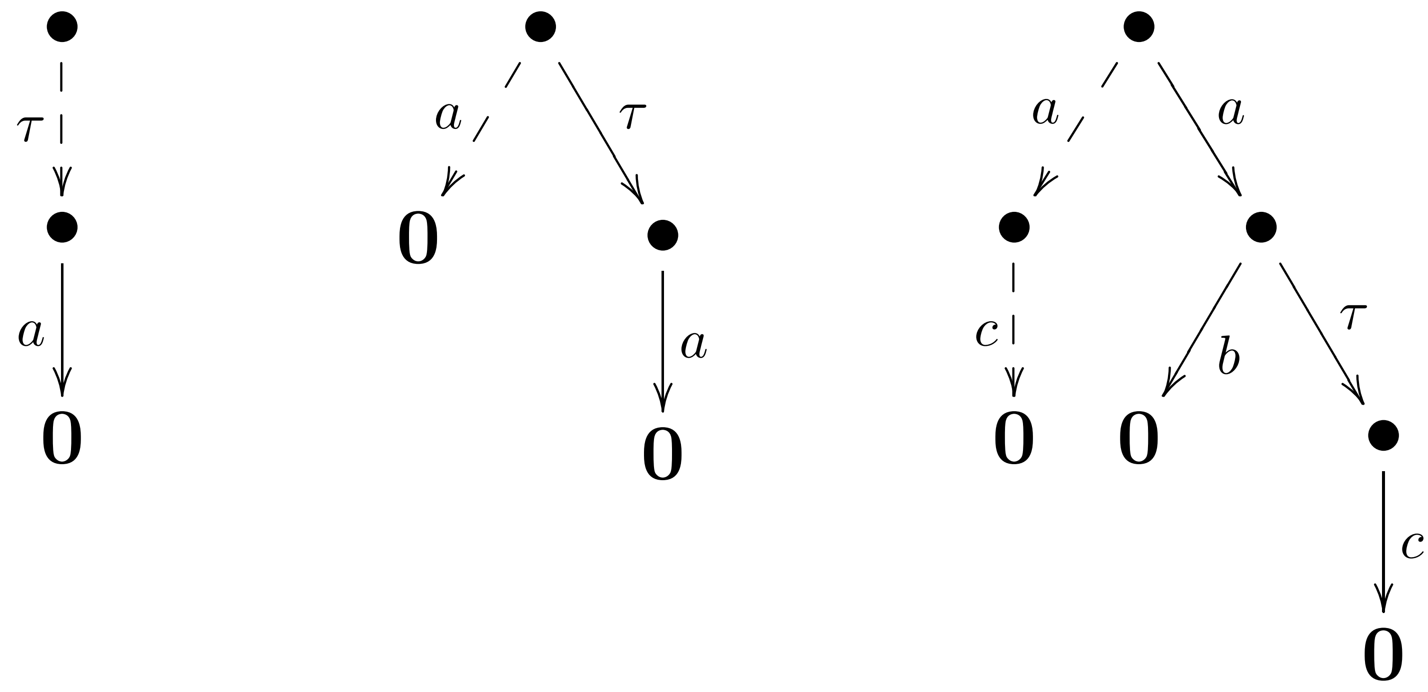
$$E \underset{\overline{c}}{\overset{a}{\rightleftarrows}} E' \underset{\overline{c}}{\overset{a}{\rightleftarrows}} E''$$

Some Inequivalences



$P = a + b \qquad Q = a + \tau.b \qquad R = \tau.a + \tau.b$

Some Equivalences



$$\tau.a \approx a \qquad a + \tau.a \approx \tau.a \qquad a.c + a.(b + \tau.c) \approx a.(b + \tau.c)$$

Some Equations

Theorem:
Let P be any process.
Let N, M any summations. Then:

1. $P \approx \tau.P$
2. $M + N + \tau.N \approx M + \tau.N$
3. $M + \alpha.P + \alpha(\tau.P + N) \approx M + \alpha(\tau.P + N)$

Congruence Properties

Proposition:
Weak bisimilarity is a process congruence, i.e.,

...

Example:

- Observe $b \approx \tau.b$!
- Check $a + b \stackrel{?}{\approx} a + \tau.b$!

Unique Solution of Equations

Theorem:

Let $\vec{X} = X_1, X_2, \ldots$ be a (possibly infinite) sequence of process variables. In the equations

$$\begin{array}{rcl} X_1 & \approx & \alpha_{11}.X_{k(11)} + \cdots + \alpha_{1n_1}.X_{k(1n_1)} \\ X_2 & \approx & \alpha_{21}.X_{k(11)} + \cdots + \alpha_{2n_1}.X_{k(2n_1)} \\ \cdots & \approx & \cdots \end{array}$$

assume that $\alpha_{ij} \neq \tau$. Then, up to $\approx$, there is a unique sequence $P_1, P_2, \ldots$ of processes which satisfies the equations.