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n-ary Semaphores [Mil99: Ex. 5.14]

$$\begin{array}{lcl} S^{(1)} & \stackrel{\text{def}}{=} & p.S_1^{(1)} \\ S_1^{(1)} & \stackrel{\text{def}}{=} & v.S^{(1)} \end{array} \qquad \begin{array}{lcl} S^{(2)} & \stackrel{\text{def}}{=} & p.S_1^{(2)} \\ S_1^{(2)} & \stackrel{\text{def}}{=} & p.S_2^{(2)} + v.S^{(2)} \\ S_2^{(2)} & \stackrel{\text{def}}{=} & v.S_1^{(2)} \end{array}$$

Proposition: $S^{(1)} \mid S^{(1)} \sim S^{(2)}$.

Proof ?

$$\mathcal{R} \stackrel{\text{def}}{=} \{ \; (S^{(1)} | S^{(1)}, S^{(2)}), \;\; (S_1^{(1)} | S_0^{(1)}, S_1^{(2)}), \;\; (S_1^{(1)} | S_1^{(1)}, S_2^{(2)}), \;\; (S_0^{(1)} | S_1^{(1)}, S_1^{(2)}) \; \}$$

Bisimulation up to $\equiv$

Definition:
A binary relation $\mathcal{S}$ over $\mathcal{P}$ is a **strong simulation up to** $\equiv$ if, whenever $P \; \mathcal{S} \; Q$:

if $p \stackrel{\mu}{\longrightarrow} p'$, then there is $q' \in Q$

such that $q \stackrel{\mu}{\longrightarrow} q'$ and $p' \equiv \mathcal{S} \equiv q'$.

Strong bisimulation up to $\equiv \ldots$

Proposition:
Let $\mathcal{S}$ be a strong simulation up to $\equiv$.
If $P \; \mathcal{S} \; Q$, then $P \sim Q$.

Bisimulation up to $\equiv$

Check !

$$\mathcal{R}_1 \stackrel{\text{def}}{=} \{ \; (S^{(1)} | S^{(1)}, S^{(2)}), \;\; (S_1^{(1)} | S_1^{(1)}, S_2^{(2)}), \;\; (S_0^{(1)} | S_1^{(1)}, S_1^{(2)}) \; \}$$

and

$$\mathcal{R}_2 \stackrel{\text{def}}{=} \{ \; (S^{(1)} | S^{(1)}, S^{(2)}), \;\; (S_1^{(1)} | S_0^{(1)}, S_1^{(2)}), \;\; (S_1^{(1)} | S_1^{(1)}, S_2^{(2)}) \; \}$$

are both strong bisimulations up to $\equiv$.

Towards Observation Equivalence

Different internal behavior should not count !

Definition: (observations / weak actions)

1. $\Rightarrow \stackrel{\text{def}}{=} \rightarrow^*$
2. - $\stackrel{\lambda}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow \stackrel{\lambda}{\longrightarrow} \Rightarrow$
   - $\stackrel{\tau}{\Longrightarrow} \stackrel{\text{def}}{=} \rightarrow^+ \quad = \quad \Rightarrow \stackrel{\tau}{\longrightarrow} \Rightarrow$
3. Let $e = \lambda_1 \cdots \lambda_n$. (If $n = 0$ then $e = \epsilon$.)
   - $\stackrel{\epsilon}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow$
   - $\stackrel{\lambda_1 \cdots \lambda_n}{\Longrightarrow} \stackrel{\text{def}}{=} \stackrel{\lambda_1}{\Longrightarrow} \cdots \stackrel{\lambda_n}{\Longrightarrow}$

Weak Simulation

Definition:
Let $\mathcal{S}$ be a binary relation over $\mathcal{P}$.
$\mathcal{S}$ is a weak simulation if, whenever $P \; \mathcal{S} \; Q$,
if $P \stackrel{e}{\Longrightarrow} P'$ then there is $Q' \in \mathcal{P}$
such that $Q \stackrel{e}{\Longrightarrow} Q'$ and $P' \; \mathcal{S} \; Q'$.

Q weakly simulates P
if there is a weak simulation $\mathcal{S}$ with $P \; \mathcal{S} \; Q$.

Example:
Prove that $Q = \tau.a.\tau.b.Q$ simulates $P = a.b.P$.

Weak Simulation (II)

Proposition:
$\mathcal{S}$ is a weak simulation **iff**, whenever $P \; \mathcal{S} \; Q$,

- if $P \rightarrow P'$ then there is $Q' \in \mathcal{P}$
  such that $Q \Rightarrow Q'$ and $P' \; \mathcal{S} \; Q'$.
- if $P \stackrel{\lambda}{\longrightarrow} P'$ then there is $Q' \in \mathcal{P}$
  such that $Q \stackrel{\lambda}{\Longrightarrow} Q'$ and $P' \; \mathcal{S} \; Q'$.

Weak Bisimulation

Definition:
. . . (* straightforward / no surprise *)

P and Q are **weakly bisimilar**,
weakly equivalent, or **observation equivalent**,
written $P \approx Q$,
if there exists a weak bisimulation $\mathcal{B}$ with $P \; \mathcal{B} \; Q$.

Proposition:

1. $\approx$ is an equivalence relation.
2. $\approx$ is itself a weak bisimulation.

Strong vs Weak

1. every strong simulation is also a weak one
2. $P \sim Q$ implies $P \approx Q$
3. see examples later on ...

Weak simulation up to $\sim$

Definition:
$\mathcal{S}$ is a weak simulation up to $\sim$
if, whenever $P \; \mathcal{S} \; Q$,

- if $P \rightarrow P'$ then there is $Q' \in \mathcal{P}$ such that $Q \Rightarrow Q'$ and $P' \sim\mathcal{S}\sim Q'$.
- if $P \stackrel{\lambda}{\longrightarrow} P'$ then there is $Q' \in \mathcal{P}$ such that $Q \stackrel{\lambda}{\Longrightarrow} Q'$ and $P' \sim\mathcal{S}\sim Q'$.

$\mathcal{S}$ is a weak bisimulation up to $\sim$
if its converse also has this property.

Weak simulation up to $\sim$ (II)

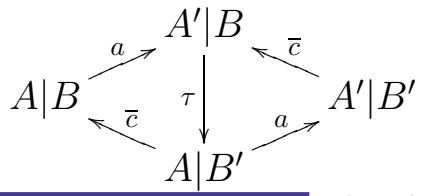
Proposition:
If $\mathcal{B}$ is a weak bisimulation up to $\sim$
and $P \; \mathcal{B} \; Q$, then $P \approx Q$.

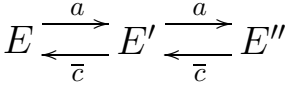
Proof ?

Example

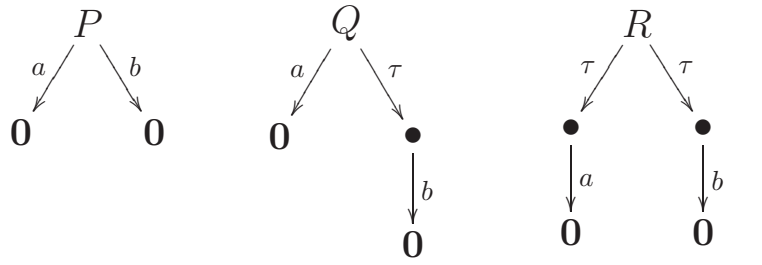
$$\begin{array}{lcll} A & \stackrel{\text{def}}{=} & a.A' & (= a.\overline{b}.A) \\ A' & \stackrel{\text{def}}{=} & \overline{b}.A & \\ B & \stackrel{\text{def}}{=} & b.B' & (= b.\overline{c}.B) \\ B' & \stackrel{\text{def}}{=} & \overline{c}.B & \end{array} \qquad \begin{array}{lcl} E & \stackrel{\text{def}}{=} & a.E' \\ E' & \stackrel{\text{def}}{=} & a.E'' + \overline{c}.E \\ E'' & \stackrel{\text{def}}{=} & \overline{c}.E' \end{array}$$

Prove that $(\nu b)(A|B) \approx E$.



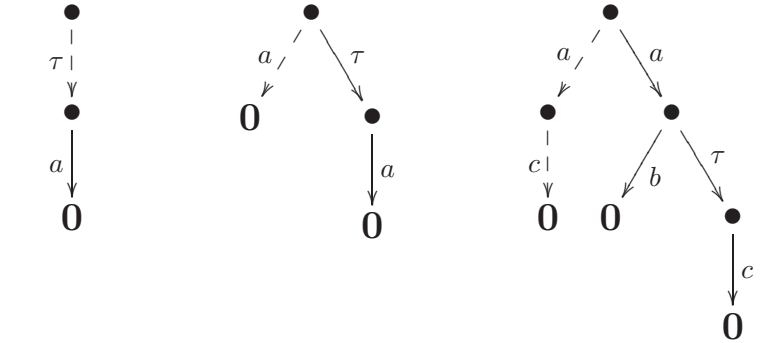


Some Inequivalences



$P = a + b$ $\quad Q = a + \tau.b$ $\quad R = \tau.a + \tau.b$

Some Equivalences



$\tau.a \approx a$ $\quad a + \tau.a \approx \tau.a$ $\quad a.c + a.(b + \tau.c) \approx a.(b + \tau.c)$

Some Equations

Theorem:
Let P be any process.
Let N, M any summations. Then:

1. $P \approx \tau.P$
2. $M + N + \tau.N \approx M + \tau.N$
3. $M + \alpha.P + \alpha(\tau.P + N) \approx M + \alpha(\tau.P + N)$

Congruence Properties

Proposition:
Weak bisimilarity is a process congruence, i.e.,
. . .

Example:

- Observe $b \approx \tau.b$!
- Check $a + b \stackrel{?}{\approx} a + \tau.b$!

Unique Solution of Equations

Theorem:

Let $\vec{X} = X_1, X_2, \ldots$ be a (possibly infinite) sequence of process variables. In the equations

$$\begin{array}{lcl} X_1 & \approx & \alpha_{11}.X_{k(11)} + \cdots + \alpha_{1n_1}.X_{k(1n_1)} \\ X_2 & \approx & \alpha_{21}.X_{k(11)} + \cdots + \alpha_{2n_1}.X_{k(2n_1)} \\ \cdots & \approx & \cdots \end{array}$$

assume that $\alpha_{ij} \neq \tau$. Then, up to $\approx$, there is a unique sequence $P_1, P_2, \ldots$ of processes which satisfies the equations.