Equivalence Notions

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Repetition of Algebraic Notions (II)

relations/functions

- composition
- comparison, containment

preorder/equivalence

- reflexivity
- symmetry
- transitivity
- comparison, containment vs fine/coarse

Repetition: Strong Simulation

Learned by heart ?

Then write it down from memory !

A binary relation $\mathcal{S}$ . . .

q simulates p if . . .

Is any simulation a preorder?

The Largest Simulation

Lemma:
If $\mathcal{S}_1$ and $\mathcal{S}_2$ are simulations, then

- $\mathcal{S}_1 \cup \mathcal{S}_2$ is also a simulation.
- $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a simulation ?
- $\mathcal{S}_1 \mathcal{S}_2$ is also a simulation ?

Definition: Let (Q, T) be a LTS.

$$\preceq \stackrel{\text{def}}{=} \bigcup \{ \ \mathcal{S} \mid \mathcal{S} \text{ is simulation over } (Q,T) \ \}$$

Fact: $\preceq$ is the largest simulation over (Q, T).

Working with Simulation

Example: Find all non-trivial simulations in
$\{(1, b, 2), (1, c, 3), (4, b, 5), (6, b, 7), (6, c, 8), (6, c, 9)\}$
How many are there ?

Trivial pairs any pairs with elements from $\{2, 3, 5, 7, 8, 9\}$ (because there are no transitions), as well as any identity on $\{1, 4, 6\}$.

Trivial simulations are those that either (0) are empty, or (1) contain only trivial pairs, or (2) contain at least one trivial pair that is not reachable from a contained non-trivial one.

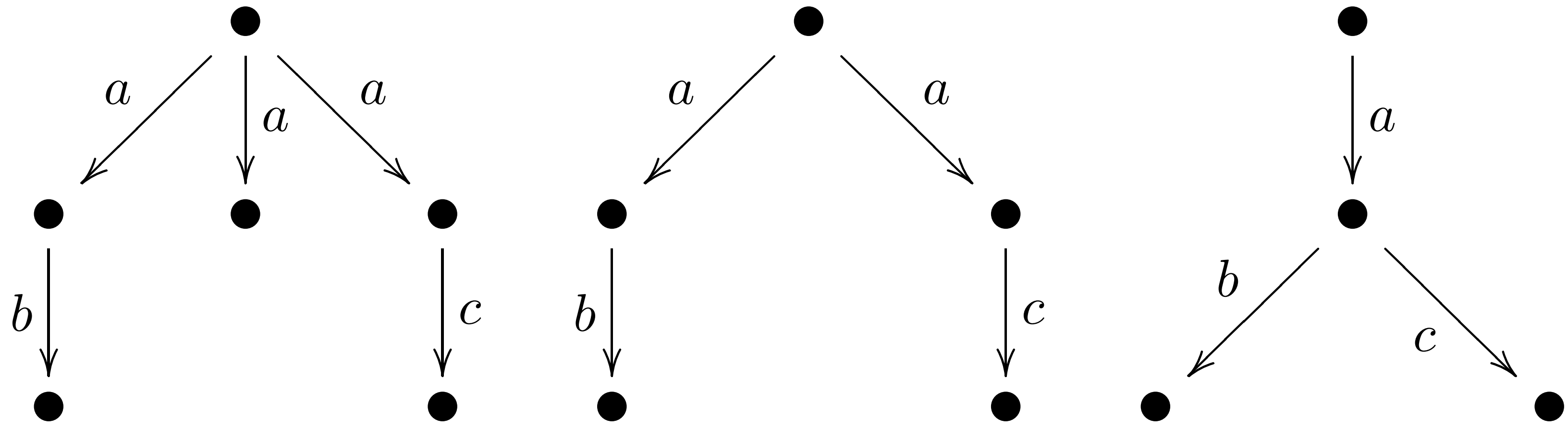
Mutual Simulation: Back and Forth

Definition:

Let (Q, T) be a LTS. Let $\{p, q\} \subseteq Q$.

p and q are **mutually similar**, written $p \gtrless q$, if there is a pair $(\mathcal{S}_1, \mathcal{S}_2)$ of simulations $\mathcal{S}_1$ and $\mathcal{S}_2$ with $p\mathcal{S}_1 q\mathcal{S}_2 p$.

Example: Mut. Sim. vs Lang. Equiv.



Mutual Simulation (II)

Proposition:

- $\gtrless$ is an equivalence relation.

Proof?

Mut. Sim. vs Lang. Equiv.

$\mathrm{Lang}(p) = \mathrm{Lang}(q)$ $\qquad$ $p \gtrless q$

$=_{\mathrm{Lang}}$ $\qquad$ $\gtrless$

Strong Bisimulation

<u>**Definition:**</u> **(learn it by heart!)**
A binary relation $\mathcal{B}$ over Q is a **strong bisimulation** over the LTS (Q, T) if both $\mathcal{B}$ and its converse $\mathcal{B}^{-1}$ are strong simulations.

p and q are **strongly bisimilar**, written $p \sim q$, if there is a strong bisimulation $\mathcal{B}$ such that $p\mathcal{B}q$.

Strong Bisimulation (II)

Proposition:

- $\sim$ is an equivalence relation.
- $\sim$ is (itself) a strong bisimulation.
- $\sim$ is the largest strong bisimulation.

Proof?

Example

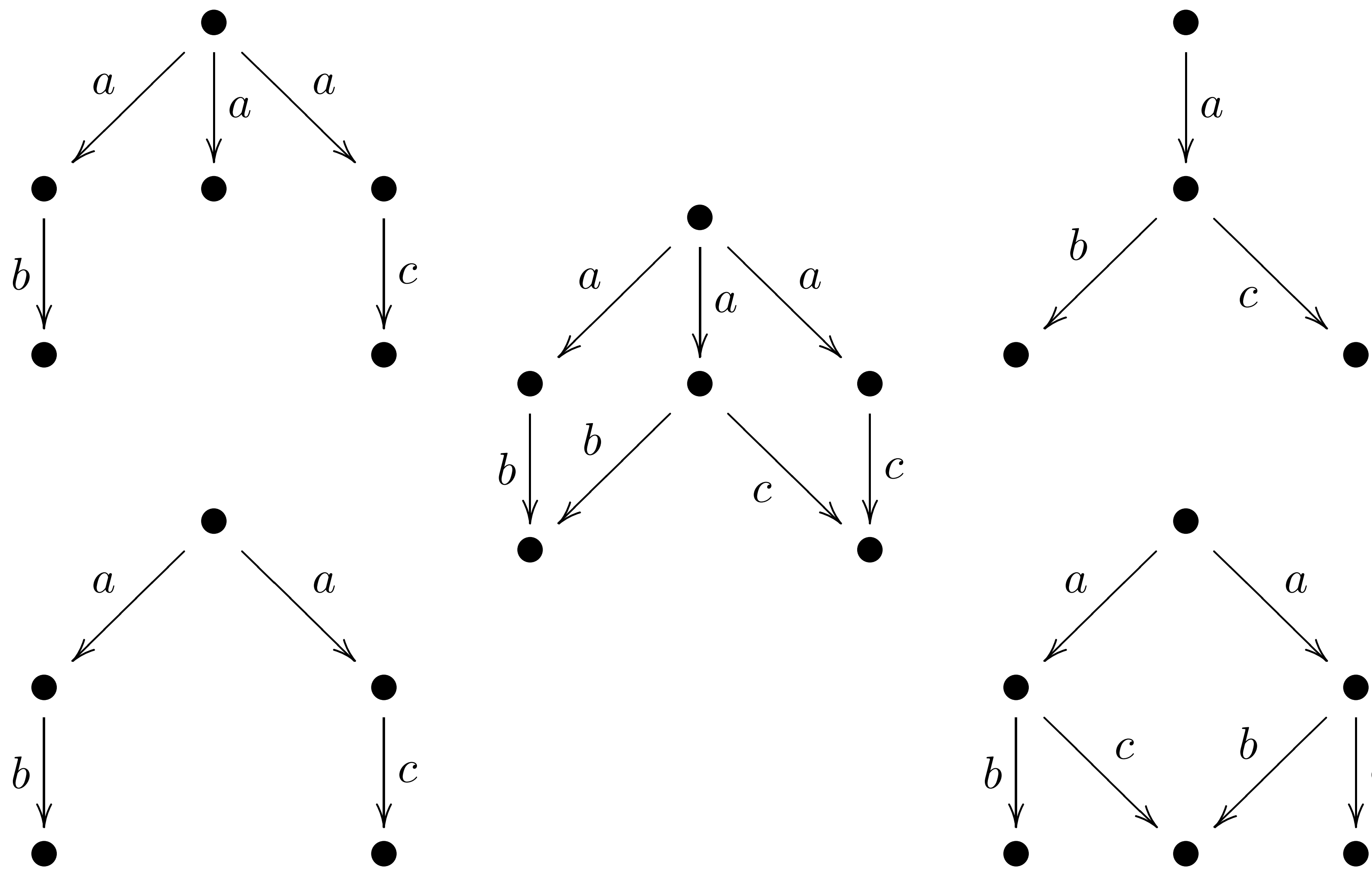
$$\{ (1,a,2),(1,a,3),(2,a,3),(2,b,1),(3,a,3),(3,b,1),$$
$$(4,a,5),(5,a,5),(5,b,6),(6,a,5),$$
$$(7,a,8),(8,a,8),(8,b,7)\ \}$$

Prove $1 \sim 4 \sim 7$.

Write out $\sim \ldots$

Minimize ?

Example: Mutual vs Bi



Isomorphism on LTS

Definition:
Let (Q_i, T_i) be two LTS over *Act* for $i \in \{1, 2\}$.

(Q_1, T_1) and (Q_2, T_2) are **isomorph(ic)**,
written $(Q_1, T_1) \cong (Q_2, T_2)$,
if there is a **bijection** f on between Q_1 and Q_2
that preserves T, i.e., $f : Q_1 \rightarrow Q_2$ with

$$q \stackrel{\alpha}{\longrightarrow} q' \quad \text{iff} \quad f(q) \stackrel{\alpha}{\longrightarrow} f(q').$$

Isomorphism on LTS (II)

Proposition:

- $\cong$ is an equivalence relation (on the domain of LTSs).

Proof?

Be careful with the interpretation of reflexivity, symmetry, and transitivity ...

Isomorphism vs Bisimulation

“Problem”:
Isomorphism compares two transition systems;
Bisimulation (at least as we have defined it) compares two states.

Redefine $\mathcal{B} \subseteq Q_1 \times Q_2$ to be a bisimulation if $\mathcal{B}$ and $\mathcal{B}^{-1}$ are simulations on their respective domains, i.e., $\mathcal{B}^{-1} \subseteq Q_2 \times Q_1$.

Redefine $\sim$ to the whole domain of LTSs.
Be careful with the interpretation of reflexivity, symmetry, and transitivity $\ldots$

Isomorphism vs Bisimulation

1. **reachability**

   $(Q_1, T_1) = (\{q_1^0, q_1^1, q_1^2\}, \{(q_1^0, a, q_1^1)\})$

   $(Q_2, T_2) = (\{q_2^0, q_2^1\}, \{(q_2^0, a, q_2^1)\})$

Isomorphism vs Bisimulation

2. **copying**

$$(Q_1, T_1) = (\{ q_1^0, q_1^1, q_1^2 \}, \{ (q_1^0, a, q_1^1), (q_1^1, b, q_1^2), (q_1^1, c, q_1^3) \})$$

$$(Q_2, T_2) =$$

$$(\{ q_1^0, q_1^1, q_1^2, q_1^3, \underline{q'^1_1, q'^2_1, q'^3_1} \},$$
$$\{ (q_2^0, a, q_2^1), (q_2^1, b, q_2^2), (q_2^1, c, q_2^3), (q_2^0, a, q'^1_2), (q'^1_2, b, q'^2_2), (q'^1_2, c, q'^3_2) \})$$

Isomorphism vs Bisimulation

3. **recursion/unfolding**

$$(Q_1, T_1) = (\{q_i \mid i \in \mathbb{N}_0\}, \{(q_i, a, q_{i+1}) \mid i \in \mathbb{N}_0\})$$
$$(Q_2, T_2) = (\{q_0\}, \qquad \{(q_0, a, q_0)\} \qquad)$$

Which is the Best Equivalence ?

language equivalence
mutual simulatity
bisimilarity
isomorphism
identity

$$= \qquad \cong \qquad \sim \qquad \gtrless \qquad =_{\mathrm{L}}$$

To be remembered: What are the intuitive distinguishing aspects between all of these notions of equivalence? ($\rightarrow$ Exam . . .)