Equivalence Notions

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Repetition of Algebraic Notions (II)

relations/functions

- composition
- comparison, containment

preorder/equivalence

- reflexivity
- symmetry
- transitivity
- comparison, containment vs fine/coarse

Repetition: Strong Simulation

Learned by heart ?

Then write it down from memory !

A binary relation $\mathcal{S}$ ...

q simulates p if ...

Is any simulation a preorder?

The Largest Simulation

Lemma:
If $\mathcal{S}_1$ and $\mathcal{S}_2$ are simulations, then

- $\mathcal{S}_1 \cup \mathcal{S}_2$ is also a simulation.
- $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a simulation ?
- $\mathcal{S}_1\mathcal{S}_2$ is also a simulation ?

Definition: Let (Q, T) be a LTS.

$$\preceq \stackrel{\text{def}}{=} \bigcup\{\ \mathcal{S} \mid \mathcal{S} \text{ is simulation over } (Q,T)\ \}$$

Fact: $\preceq$ is the largest simulation over (Q, T).

Working with Simulation

Example: Find all non-trivial simulations in
$\{(1,b,2),(1,c,3),(4,b,5),(6,b,7),(6,c,8),(6,c,9)\}$
How many are there ?

Trivial pairs any pairs with elements from $\{2,3,5,7,8,9\}$ (because there are no transitions), as well as any identity on $\{1,4,6\}$.

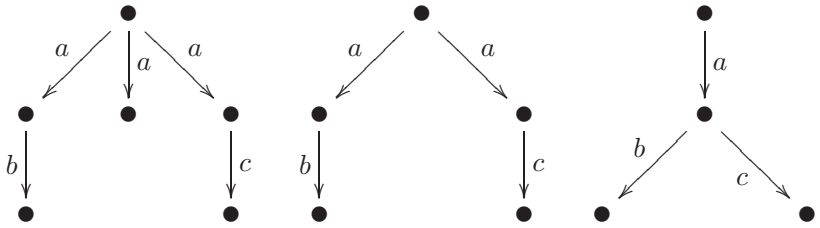
Trivial simulations are those that either (0) are empty, or (1) contain only trivial pairs, or (2) contain at least one trivial pair that is not reachable from a contained non-trivial one.

Mutual Simulation: Back and Forth

Definition:
Let (Q,T) be a LTS. Let $\{p,q\} \subseteq Q$.

p and q are **mutually similar**, written $p \geqslant q$, if there is a pair $(\mathcal{S}_1,\mathcal{S}_2)$ of simulations $\mathcal{S}_1$ and $\mathcal{S}_2$ with $p\mathcal{S}_1 q \mathcal{S}_2 p$.

Example: Mut. Sim. vs Lang. Equiv.



Mutual Simulation (II)

Proposition:

- $\gtrless$ is an equivalence relation.

Proof?

Mut. Sim. vs Lang. Equiv.

$\mathrm{Lang}(p) = \mathrm{Lang}(q)$ $\qquad$ $p \gtrless q$

$=_{\mathrm{Lang}}$ $\qquad$ $\gtrless$

Strong Bisimulation

Definition: **(learn it by heart!)**
A binary relation $\mathcal{B}$ over Q is a **strong bisimulation** over the LTS (Q, T) if both $\mathcal{B}$ and its converse $\mathcal{B}^{-1}$ are strong simulations.

p and q are **strongly bisimilar**, written $p \sim q$, if there is a strong bisimulation $\mathcal{B}$ such that $p\mathcal{B}q$.

Strong Bisimulation (II)

Proposition:

- $\sim$ is an equivalence relation.
- $\sim$ is (itself) a strong bisimulation.
- $\sim$ is the largest strong bisimulation.

Proof?

Example

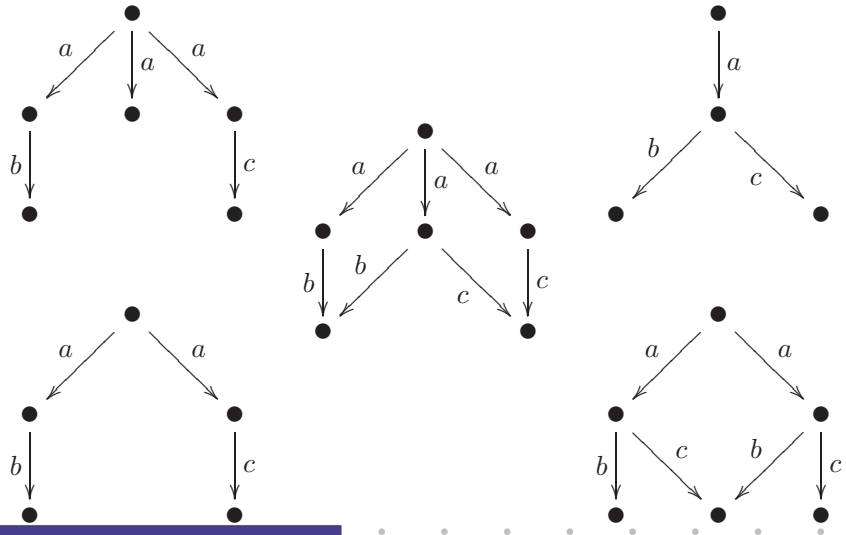
$\{ (1, a, 2), (1, a, 3), (2, a, 3), (2, b, 1), (3, a, 3), (3, b, 1),$
$(4, a, 5), (5, a, 5), (5, b, 6), (6, a, 5),$
$(7, a, 8), (8, a, 8), (8, b, 7) \}$

Prove $1 \sim 4 \sim 7$.

Write out $\sim \ldots$

Minimize ?

Example: Mutual vs Bi



Isomorphism on LTS

Definition:

Let (Q_i, T_i) be two LTS over *Act* for $i \in \{1, 2\}$.

(Q_1, T_1) and (Q_2, T_2) are **isomorph(ic)**,
written $(Q_1, T_1) \cong (Q_2, T_2)$,
if there is a **bijection** f on between Q_1 and Q_2
that preserves T, i.e., $f : Q_1 \to Q_2$ with

$$q \xrightarrow{\alpha} q' \quad \text{iff} \quad f(q) \xrightarrow{\alpha} f(q').$$

Isomorphism on LTS (II)

Proposition:

- $\cong$ is an equivalence relation (on the domain of LTSs).

Proof?

Be careful with the interpretation of reflexivity, symmetry, and transitivity ...

Isomorphism vs Bisimulation

"Problem":
Isomorphism compares two transition systems;
Bisimulation (at least as we have defined it) compares two states.

Redefine $\mathcal{B} \subseteq Q_1 \times Q_2$ to be a bisimulation if $\mathcal{B}$ and $\mathcal{B}^{-1}$ are simulations on their respective domains, i.e., $\mathcal{B}^{-1} \subseteq Q_2 \times Q_1$.

Redefine $\sim$ to the whole domain of LTSs.
Be careful with the interpretation of reflexivity, symmetry, and transitivity ...

Isomorphism vs Bisimulation

1. **reachability**

$(Q_1, T_1) = (\{q_1^0, q_1^1, q_1^2\}, \{(q_1^0, a, q_1^1)\})$

$(Q_2, T_2) = (\{q_2^0, q_2^1\}, \{(q_2^0, a, q_2^1)\})$

Isomorphism vs Bisimulation

2. **copying**

$(Q_1, T_1) = (\{\ q_1^0, q_1^1, q_1^2\ \},$
$\{\ (q_1^0, a, q_1^1), (q_1^1, b, q_1^2), (q_1^1, c, q_1^3)\ \}\ )$

$(Q_2, T_2) =$
$(\{\ q_1^0, q_1^1, q_1^2, q_1^3, \underline{q'^1_1, q'^2_1, q'^3_1}\ \},$
$\{\ (q_2^0, a, q_2^1), (q_2^1, b, q_2^2), (q_2^1, c, q_2^3),$
$(q_2^0, a, q'^1_2), (q'^1_2, b, q'^2_2), (q'^1_2, c, q'^3_2)\ \}\ )$

Isomorphism vs Bisimulation

3. **recursion/unfolding**

$$(Q_1, T_1) = (\{q_i \mid i \in \mathbb{N}_0\}, \{(q_i, a, q_{i+1}) \mid i \in \mathbb{N}_0\})$$
$$(Q_2, T_2) = (\{q_0\}, \{(q_0, a, q_0)\})$$

Which is the Best Equivalence ?

language equivalence
mutual simulatity
bisimilarity
isomorphism
identity

$$= \quad \cong \quad \sim \quad \geqslant \quad =_{\mathrm{L}}$$

To be remembered: What are the intuitive distinguishing aspects between all of these notions of equivalence? ($\rightarrow$ Exam . . .)