Foundations of Programming

– Concurrency –

Session 9 – April 15, 2002

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Session 9: CCS

- repetition of algebraic notions
- equivalence on transition systems
- simulation
- strong bisimulation
- weak bisimulation

Repetition of Algebraic Notions

relations/functions

- composition
- comparison, containment

preorder/equivalence

- reflexivity
- symmetry
- transitivity
- kernel of a (reflexive) preorder
- comparison, containment vs fine/coarse
- congruence

Tea ? Coffee ?

Compare p_0 and q_0 in the following LTS:

$$\{ \ (p_0, \mathsf{2c}, p_1), (p_1, \mathsf{2c}, p_2), (p_1, \overline{\mathsf{tea}}, p_0), (p_2, \overline{\mathsf{coffee}}, p_0),$$
$$(q_0, \mathsf{2c}, q_1), (q_0, \mathsf{2c}, q_1'), (q_1, \mathsf{2c}, q_2), (q_1', \overline{\mathsf{tea}}, q_0), (q_2, \overline{\mathsf{coffee}}, q_0), \ \}$$

In which sense are they different or equivalent?

Equivalence on LTS ?

Example: Compare p_0 and q_0 in

$$\{ \quad (p_0, a, p_1), (p_1, b, p_2), (p_1, c, p_3), \\ (q_0, a, q_1), (q_0, a, q_1'), (q_1, b, q_2), (q_1', c, q_3) \quad \}$$

Induce simulation of paths
through step-by-step simulation of actions ...

(Strong) Simulation on LTS

Definition: **(learn it by heart!)**
Let $(\mathcal{Q}, \mathcal{T})$ be an LTS.

1. Let $\mathcal{S}$ be a binary relation over $\mathcal{Q}$.
   $\mathcal{S}$ is a **(strong) simulation** over $(\mathcal{Q}, \mathcal{T})$ if, whenever $p \mathrel{\mathcal{S}} q$,
   $\boxed{\text{if } p \stackrel{\alpha}{\longrightarrow} p' \text{ then there is } q' \in \mathcal{Q} \text{ such that } q \stackrel{\alpha}{\longrightarrow} q' \text{ and } p' \mathrel{\mathcal{S}} q'}$.

2. q **(strongly) simulates** p, written $p \preceq q$,
   if there is a (strong) simulation $\mathcal{S}$ such that $p \mathrel{\mathcal{S}} q$.

The relation $\preceq$ is sometimes called *similarity*.

Properties of Simulations

Lemma:

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are simulations, then

- $\mathcal{S}_1 \cup \mathcal{S}_2$ is also a simulation.
- $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a simulation ?
- $\mathcal{S}_1\mathcal{S}_2$ is also a simulation ?

Definition: Let $(\mathcal{Q}, \mathcal{T})$ be a LTS.

$$\preceq \stackrel{\text{def}}{=} \bigcup\{\ \mathcal{S} \mid \mathcal{S} \text{ is simulation over } (\mathcal{Q}, \mathcal{T})\ \}$$

Lemma:

- $\preceq$ is the largest simulation over $(\mathcal{Q}, \mathcal{T})$.
- $\preceq$ is a reflexive preorder over $\mathcal{Q} \times \mathcal{Q}$.

Working with Simulation

BTW, is any simulation a preorder?

What do we do with simulations?

- exhibiting a simulation:
  - "*guessing*" a simulation $\mathcal{S}$ that contains (p, q)
  - "*generating*" a simulation: do it algorithmically !
    $\rightarrow$ bisimulation-checking algorithms (CWB)
    $\rightarrow$ decidability ?
- checking a simulation:
  check that a given relation $\mathcal{S}$ is in fact a simulation.

Home-Working with Simulation

Example: Find all non-trivial simulations in

$$\{(1,b,2),(1,c,3),(4,b,5),(6,b,7),(6,c,8),(6,c,9)\}$$

How many are there ?

Trivial pairs are any pairs with elements from $\{2,3,5,7,8,9\}$ (because there are no transitions),
as well as any identity on $\{1,4,6\}$.

Trivial simulations are those that either
(0) are empty, or
(1) contain only trivial pairs, or
(2) contain at least one trivial pair that is not reachable from a contained non-trivial one.

Towards Equivalence

Simulation is only a preorder,
thus it allows us to *distinguish* states.

We want instead an equivalence,
which would allow us to *equate* states.

The mathematical way: just take the “kernel”

$$p = q \qquad \text{if} \qquad p < q \qquad \text{and} \qquad q < p$$

However, there are two different natural candidates !

- mutual simulation
- bisimulation

Mutual Simulation: Back and Forth

Definition:
Let $(\mathcal{Q}, \mathcal{T})$ be a LTS. Let $\{p, q\} \subseteq \mathcal{Q}$.

p and q are **mutually similar**, written $p \gtrless q$,
if there is a pair $(\mathcal{S}_1, \mathcal{S}_2)$ of simulations $\mathcal{S}_1$ and $\mathcal{S}_2$
with $p\ \mathcal{S}_1\ q\ \mathcal{S}_2\ p$ (i.e., with $p\ \mathcal{S}_1\ q$ and $q\ \mathcal{S}_2\ p$).

Mutual Simulation (II)

Proposition:

- $\gtrless$ is an equivalence relation.

Proof?

Typical research-craftsmen work ...

(Strong) Bisimulation

Definition: **(learn it by heart!)**
A binary relation $\mathcal{B}$ over $\mathcal{Q}$ is
a **(strong) bisimulation** over the LTS $(\mathcal{Q}, \mathcal{T})$
if $\boxed{\text{both } \mathcal{B} \text{ and its converse } \mathcal{B}^{-1} \text{ are (strong) simulations}}$.

p and q are **(strongly) bisimilar**, written $p \sim q$,
if there is a (strong) bisimulation $\mathcal{B}$ such that $p \; \mathcal{B} \; q$.

$$\sim \stackrel{\text{def}}{=} \bigcup \{ \; \mathcal{B} \mid \mathcal{B} \text{ is bisimulation over } (\mathcal{Q}, \mathcal{T}) \; \}$$

(Strong) Bisimulation (II)

Proposition:

- $\sim$ is an equivalence relation.
- $\sim$ is (itself) a (strong) bisimulation.
- $\sim$ is the largest (strong) bisimulation.

Proof?

Again, typical research-craftsmen work . . .

Example

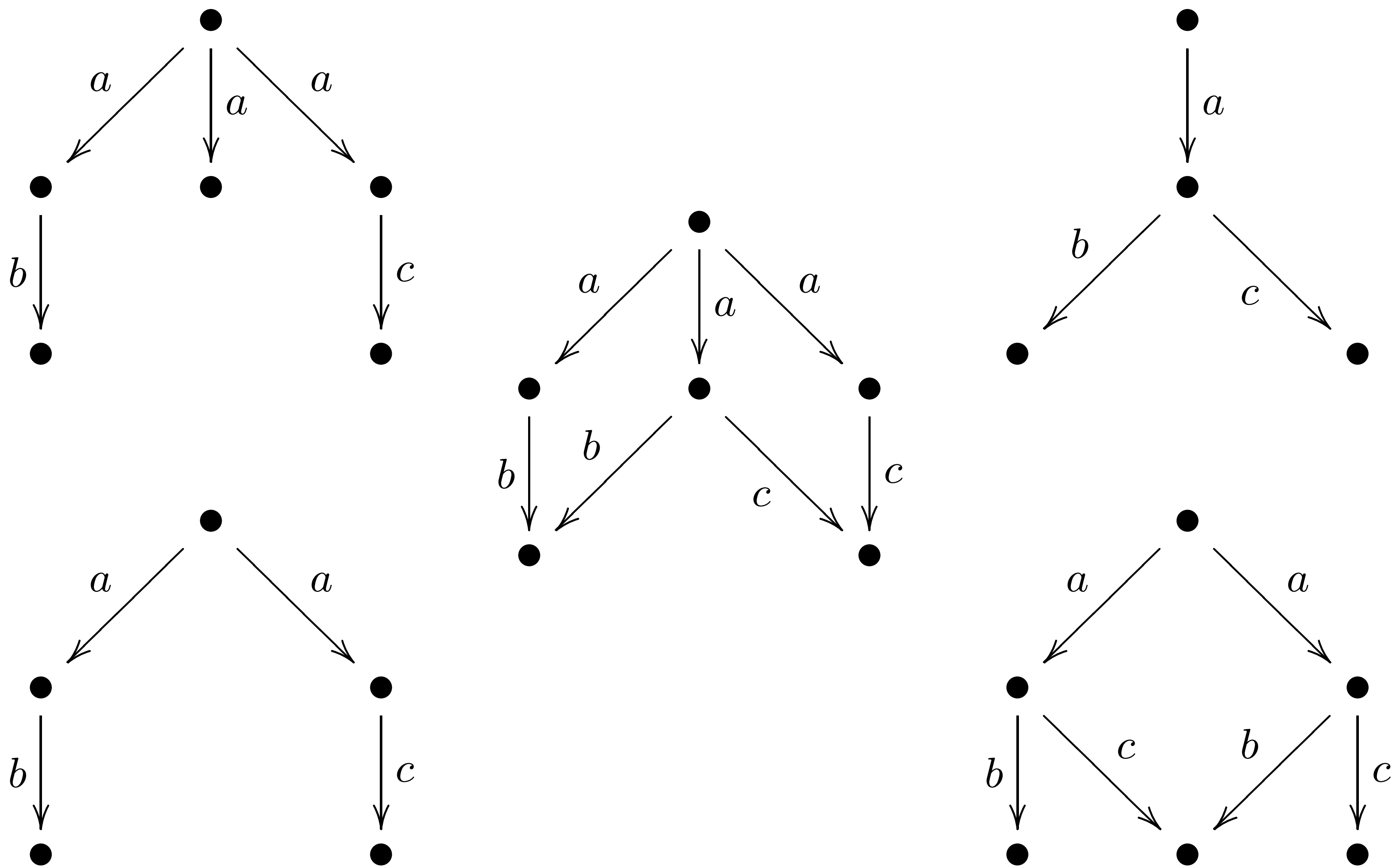
$$\{\,(1,a,2),(1,a,3),(2,a,3),(2,b,1),(3,a,3),(3,b,1),$$
$$(4,a,5),(5,a,5),(5,b,6),(6,a,5),$$
$$(7,a,8),(8,a,8),(8,b,7)\,\}$$

Prove $1 \sim 4 \sim 6 \sim 7$.

Write out $\sim \ldots$

Minimization ?!

Example: Mutual vs Bi



Towards Observation Equivalence

Let us assume that our LTSs may dispose of a single distinguished *internal action* symbol, say: τ, as is the case for our language of concurrent process expressions. Then:

“Different internal behavior” should “not count” !

Definition: (observations / weak actions)

1. $\Rightarrow \stackrel{\text{def}}{=} \rightarrow^{*}$
2. $\stackrel{\lambda}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow \stackrel{\lambda}{\longrightarrow} \Rightarrow$

Weak Simulation

Definition:

$\mathcal{S}$ is a weak simulation **iff**, whenever $p\ \mathcal{S}\ q$,

- if $p \to p'$ then there is q' such that $q \Rightarrow q'$ and $p'\ \mathcal{S}\ q'$.
- if $p \xrightarrow{\lambda} p'$ then there is q' such that $q \stackrel{\lambda}{\Longrightarrow} q'$ and $p'\ \mathcal{S}\ q'$.

q **weakly simulates** p,
if there is a weak simulation $\mathcal{S}$ such that $p\ \mathcal{S}\ q$.

Example:

Prove that $Q = \tau.a.\tau.b.Q$ simulates $P = a.b.P$.

Weak Bisimulation

Definition:

. . . (* straightforward / no surprise *)

p and q are **weakly bisimilar**,
weakly equivalent, or **observation equivalent**,
written $p \approx q$,
if there exists a weak bisimulation $\mathcal{B}$ with $p \mathrel{\mathcal{B}} q$.

$$\approx \stackrel{\mathrm{def}}{=} \bigcup \{ \ \mathcal{B} \mid \mathcal{B} \text{ is weak bisimulation over } (\mathcal{Q}, \mathcal{T}) \ \}$$

Proposition:

1. $\approx$ is an equivalence relation.
2. $\approx$ is itself a weak bisimulation.

Strong vs Weak

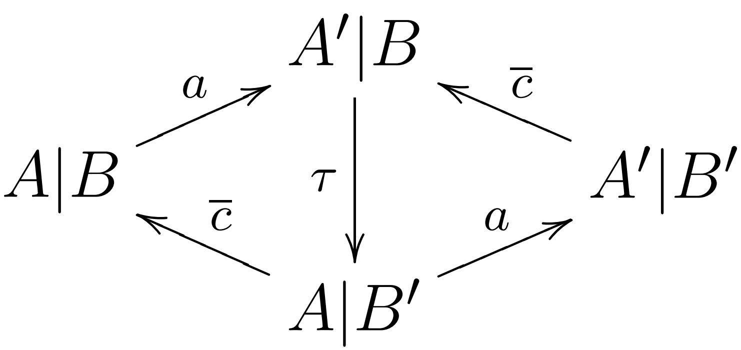
1. every strong simulation is also a weak one
2. $p \sim q$ implies $p \approx q$
3. see examples later on $\ldots$

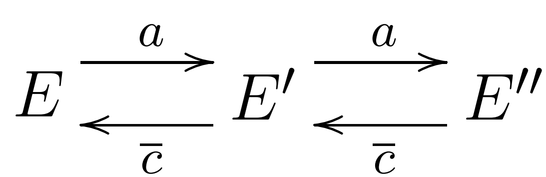
Example

$$
\begin{aligned}
A &\stackrel{\text{def}}{=} a.A' \quad (= a.\overline{b}.A) \\
A' &\stackrel{\text{def}}{=} \overline{b}.A \\
B &\stackrel{\text{def}}{=} b.B' \quad (= b.\overline{c}.B) \\
B' &\stackrel{\text{def}}{=} \overline{c}.B
\end{aligned}
$$

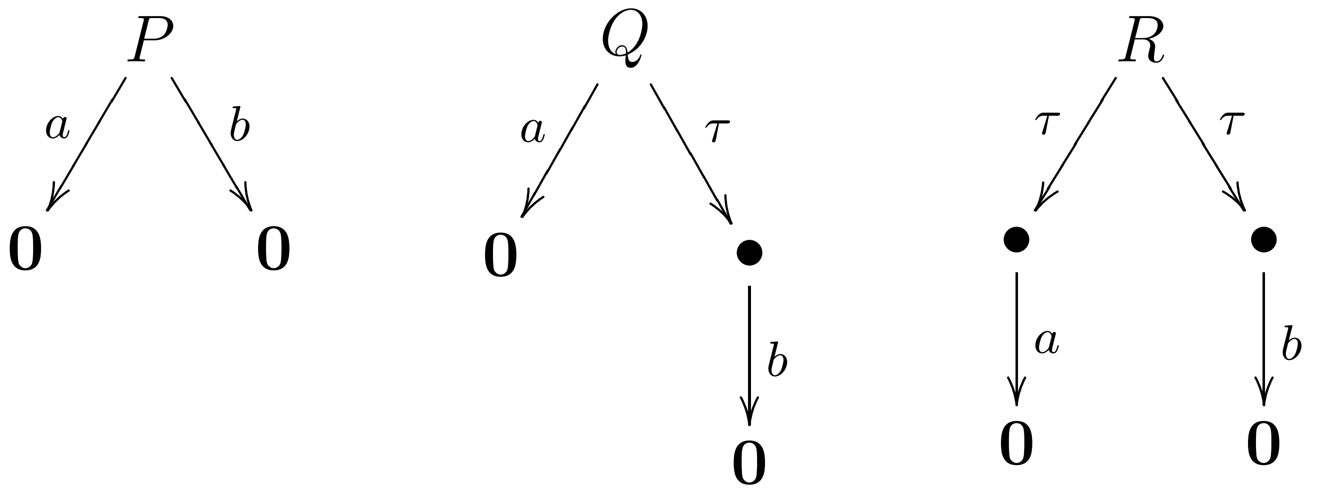
$$
\begin{aligned}
E &\stackrel{\text{def}}{=} a.E' \\
E' &\stackrel{\text{def}}{=} a.E'' + \overline{c}.E \\
E'' &\stackrel{\text{def}}{=} \overline{c}.E'
\end{aligned}
$$

Prove that $(\boldsymbol{\nu} b)(A|B) \approx E$.



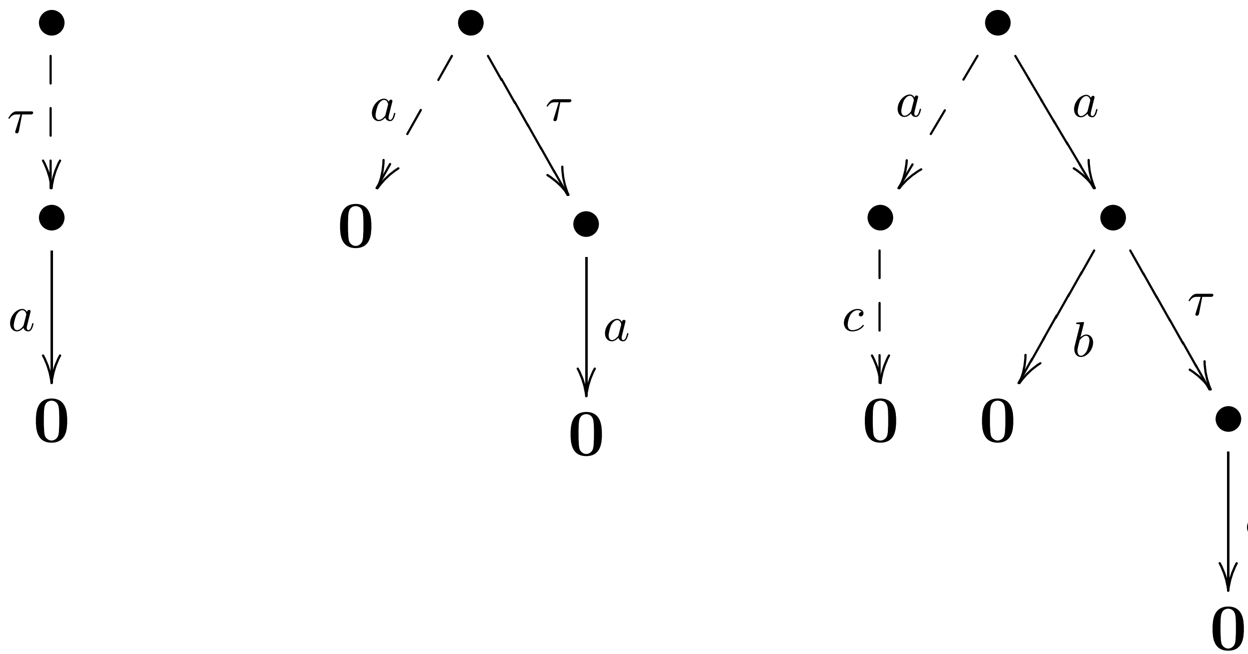


Some Inequivalences



$P = a + b$ $\qquad$ $Q = a + \tau.b$ $\qquad$ $R = \tau.a + \tau.b$

Some Equivalences



$\tau.a \approx a$

$a + \tau.a \approx \tau.a$

$$a.c + a.(b + \tau.c) \approx a.(b + \tau.c)$$

Some Equations

Theorem:
Let P be any process.
Let N, M any summations. Then:

1. $P \approx \tau.P$
2. $M + N + \tau.N \approx M + \tau.N$
3. $M + \alpha.P + \alpha(\tau.P + N) \approx M + \alpha(\tau.P + N)$