Foundations of Programming

– Concurrency –

Session 9 – April 15, 2002

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Session 9: CCS

- repetition of algebraic notions
- equivalence on transition systems
- simulation
- strong bisimulation
- weak bisimulation

Repetition of Algebraic Notions

lations/functions

- ☐ composition
- ☐ comparison, containment

eorder/equivalence

- ☐ reflexivity
- ☐ symmetry
- ☐ transitivity
- ☐ kernel of a (reflexive) preorder
- ☐ comparison, containment vs fine/coarse
- ☐ congruence

Tea ? Coffee ?

Compare p_0 and q_0 in the following LTS:

$\{\ (p_0, \mathsf{2c}, p_1), (p_1, \mathsf{2c}, p_2), (p_1, \overline{\mathsf{tea}}, p_0), (p_2, \overline{\mathsf{coffee}}, p_0),$
$(q_0, \mathsf{2c}, q_1), (q_0, \mathsf{2c}, q_1'), (q_1, \mathsf{2c}, q_2), (q_1', \overline{\mathsf{tea}}, q_0), (q_2, \overline{\mathsf{coffee}}, q_0),\ \}$

In which sense are they different or equivalent?

Equivalence on LTS ?

xample: Compare p_0 and q_0 in

$\{ \quad (p_0, a, p_1), (p_1, b, p_2), (p_1, c, p_3),$
$\quad (q_0, a, q_1), (q_0, a, q_1'), (q_1, b, q_2), (q_1', c, q_3) \quad \}$

nduce simulation of paths
hrough step-by-step simulation of actions . . .

(Strong) Simulation on LTS

Definition: **(learn it by heart!)**
Let $(\mathcal{Q}, \mathcal{T})$ be an LTS.

1. Let $\mathcal{S}$ be a binary relation over $\mathcal{Q}$.
   $\mathcal{S}$ is a **(strong) simulation** over $(\mathcal{Q}, \mathcal{T})$ if, whenever $p \mathcal{S} q$,

   if $p \stackrel{\alpha}{\longrightarrow} p'$ then there is $q' \in \mathcal{Q}$ such that $q \stackrel{\alpha}{\longrightarrow} q'$ and $p' \mathcal{S} q$

2. q **(strongly) simulates** p, written $p \preceq q$,
   if there is a (strong) simulation $\mathcal{S}$ such that $p \mathcal{S} q$.

The relation $\preceq$ is sometimes called *similarity*.

Properties of Simulations

emma:

$\mathcal{S}_1$ and $\mathcal{S}_2$ are simulations, then

- $\mathcal{S}_1 \cup \mathcal{S}_2$ is also a simulation.
- $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a simulation ?
- $\mathcal{S}_1\mathcal{S}_2$ is also a simulation ?

efinition: Let $(\mathcal{Q}, \mathcal{T})$ be a LTS.

$$\preceq \stackrel{\text{def}}{=} \bigcup\{\ \mathcal{S} \mid \mathcal{S} \text{ is simulation over } (\mathcal{Q}, \mathcal{T})\ \}$$

emma:

- $\preceq$ is the largest simulation over $(\mathcal{Q}, \mathcal{T})$.
- $\preceq$ is a reflexive preorder over $\mathcal{Q} \times \mathcal{Q}$.

Working with Simulation

BTW, is any simulation a preorder?

What do we do with simulations?

- exhibiting a simulation:
  - "*guessing*" a simulation $\mathcal{S}$ that contains (p, q)
  - "*generating*" a simulation: do it algorithmically !
    → bisimulation-checking algorithms (CWB)
    → decidability ?
- checking a simulation:
  check that a given relation $\mathcal{S}$ is in fact a simulation.

Home-Working with Simulation

xample: Find all non-trivial simulations in

$$\{(1, b, 2), (1, c, 3), (4, b, 5), (6, b, 7), (6, c, 8), (6, c, 9)\}$$

ow many are there ?

rivial pairs are any pairs with elements from $\{2, 3, 5, 7, 8, 9\}$
ecause there are no transitions),
s well as any identity on $\{1, 4, 6\}$.

rivial simulations are those that either
0) are empty, or
1) contain only trivial pairs, or
2) contain at least one trivial pair that is not reachable from a
ontained non-trivial one.

Towards Equivalence

Simulation is only a preorder,
thus it allows us to *distinguish* states.

We want instead an equivalence,
which would allow us to *equate* states.

The mathematical way: just take the "kernel"

$$p = q \quad \text{if} \quad p < q \quad \text{and} \quad q < p$$

However, there are two different natural candidates !

- ☐ mutual simulation
- ☐ bisimulation

Mutual Simulation: Back and Forth

efinition:
et $(\mathcal{Q}, \mathcal{T})$ be a LTS. Let $\{p, q\} \subseteq \mathcal{Q}$.

and q are **mutually similar**, written $p \gtrless q$,
there is a pair $(\mathcal{S}_1, \mathcal{S}_2)$ of simulations $\mathcal{S}_1$ and $\mathcal{S}_2$
ith $p\ \mathcal{S}_1\ q\ \mathcal{S}_2\ p$ (i.e., with $p\ \mathcal{S}_1\ q$ and $q\ \mathcal{S}_2\ p$).

Mutual Simulation (II)

Proposition:

- $\gtrless$ is an equivalence relation.

Proof?

Typical research-craftsmen work . . .

(Strong) Bisimulation

efinition: **(learn it by heart!)**
binary relation $\mathcal{B}$ over $\mathcal{Q}$ is
(strong) bisimulation over the LTS $(\mathcal{Q}, \mathcal{T})$
both $\mathcal{B}$ and its converse $\mathcal{B}^{-1}$ are (strong) simulations.

and q are **(strongly) bisimilar**, written $p \sim q$,
there is a (strong) bisimulation $\mathcal{B}$ such that $p \mathcal{B} q$.

$$\sim \stackrel{\text{def}}{=} \bigcup\{ \mathcal{B} \mid \mathcal{B} \text{ is bisimulation over } (\mathcal{Q}, \mathcal{T}) \}$$

(Strong) Bisimulation (II)

Proposition:

- ☐ $\sim$ is an equivalence relation.
- ☐ $\sim$ is (itself) a (strong) bisimulation.
- ☐ $\sim$ is the largest (strong) bisimulation.

Proof?

Again, typical research-craftsmen work . . .

Example

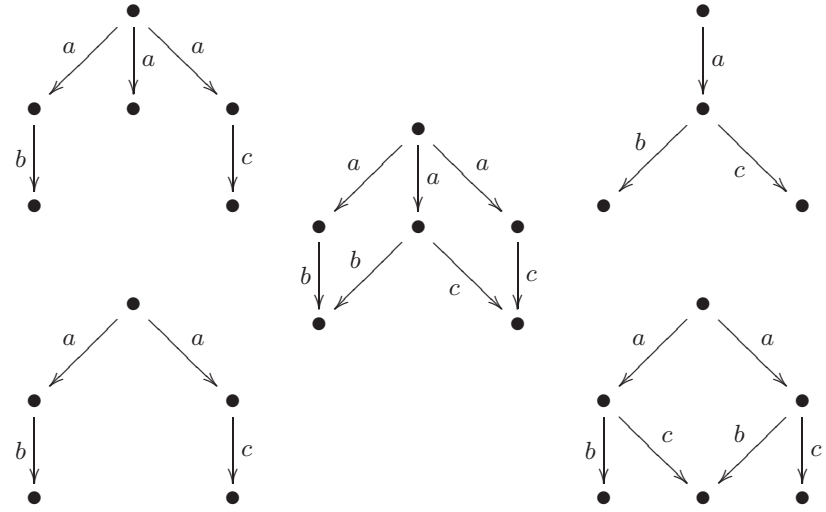
$$\{\,(1,a,2),(1,a,3),(2,a,3),(2,b,1),(3,a,3),(3,b,1),$$
$$(4,a,5),(5,a,5),(5,b,6),(6,a,5),$$
$$(7,a,8),(8,a,8),(8,b,7)\,\}$$

rove $1 \sim 4 \sim 6 \sim 7$.

/rite out $\sim \ldots$

inimization ?!

Example: Mutual vs Bi



Towards Observation Equivalence

et us assume that our LTSs may dispose of a single istinguished *internal action* symbol, say: τ, as is the case for ur language of concurrent process expressions. Then:

"Different internal behavior" should "not count" !

efinition: (observations / weak actions)

1. $\Rightarrow \stackrel{\text{def}}{=} \rightarrow^*$
2. $\stackrel{\lambda}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow \stackrel{\lambda}{\longrightarrow} \Rightarrow$

Weak Simulation

Definition:

$\mathcal{S}$ is a weak simulation **iff**, whenever $p \mathcal{S} q$,

- if $p \rightarrow p'$ then there is q' such that $q \Rightarrow q'$ and $p' \mathcal{S} q'$.
- if $p \stackrel{\lambda}{\longrightarrow} p'$ then there is q' such that $q \stackrel{\lambda}{\Longrightarrow} q'$ and $p' \mathcal{S} q'$.

q **weakly simulates** p,
if there is a weak simulation $\mathcal{S}$ such that $p \mathcal{S} q$.

Example:

Prove that $Q = \tau.a.\tau.b.Q$ simulates $P = a.b.P$.

Weak Bisimulation

efinition:

. . (* straightforward / no surprise *)

and q are **weakly bisimilar**,
eakly equivalent, or **observation equivalent**,
ritten $p \approx q$,
there exists a weak bisimulation $\mathcal{B}$ with $p \; \mathcal{B} \; q$.

$$\approx \stackrel{\text{def}}{=} \bigcup\{ \; \mathcal{B} \mid \mathcal{B} \text{ is weak bisimulation over } (\mathcal{Q}, \mathcal{T}) \; \}$$

roposition:

. $\approx$ is an equivalence relation.

2. $\approx$ is itself a weak bisimulation.

Strong vs Weak

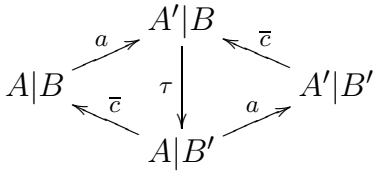
1. every strong simulation is also a weak one
2. $p \sim q$ implies $p \approx q$
3. see examples later on . . .

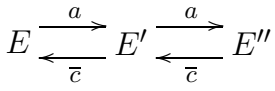
Example

$$A \stackrel{\text{def}}{=} a.A' \quad (= a.\overline{b}.A)$$
$$A' \stackrel{\text{def}}{=} \overline{b}.A$$
$$B \stackrel{\text{def}}{=} b.B' \quad (= b.\overline{c}.B)$$
$$B' \stackrel{\text{def}}{=} \overline{c}.B$$

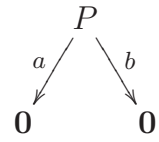
$$E \stackrel{\text{def}}{=} a.E'$$
$$E' \stackrel{\text{def}}{=} a.E'' + \overline{c}.E$$
$$E'' \stackrel{\text{def}}{=} \overline{c}.E'$$

Prove that $(\boldsymbol{\nu} b)(A|B) \approx E$.

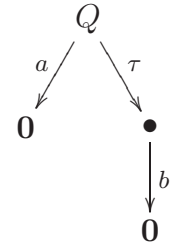




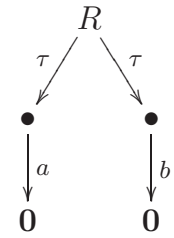
Some Inequivalences



$P = a + b$



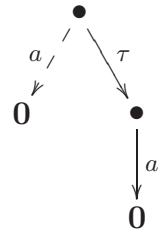
$Q = a + \tau.b$



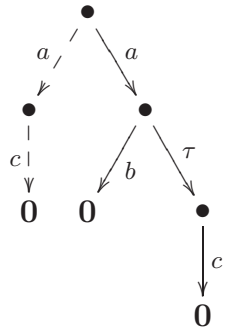
$R = \tau.a + \tau.b$

Some Equivalences

$\tau.a \approx a$



$a + \tau.a \approx \tau.a$



$a.c + a.(b + \tau.c)$
$\approx a.(b + \tau.c)$

Some Equations

Theorem:
Let P be any process.
Let N, M any summations. Then:

1. $P \approx \tau.P$
2. $M + N + \tau.N \approx M + \tau.N$
3. $M + \alpha.P + \alpha(\tau.P + N) \approx M + \alpha(\tau.P + N)$