Fondements de la programmation
Exercice série 2: Domaines Sémantiques

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Ex. 1 Soient les cpo's D et E. Montrer que, par la definition de fonction continue 3.2, si $f : D \to E$ est continue alors f est monotone ($2 \Rightarrow 1$).

Solution: We want to show that

$$x \sqsubseteq_D y \Rightarrow f(x) \sqsubseteq_E f(y).$$

Let $x, y \in D$ with $x \sqsubseteq_D y$. The set $\{x, y\}$ is a finite chain such that

$$\sqcup_D\{x, y\} = y.$$

Since f is continuos then

$$f(\sqcup_D\{x, y\}) = f(y) = \sqcup_E\{f(x), f(y)\}$$

which implies that $f(x) \sqsubseteq_E f(y)$.

Ex. 2 Soient les cpo's D et E telles que toute chaîne est stationnaire, ç.à.d.

$$\forall \langle x_i \rangle_{i\in\omega} \subseteq D : x_i \sqsubseteq x_{i+1} \Rightarrow \exists k \in \omega : x_k = \sqcup \langle x_i \rangle_{i\in\omega}$$

Montrer que si $f : D \to E$ est monotone alors f est continue ($1 \Rightarrow 2$).

Solution: We want to show that

$$\sqcup_E \langle f(x_i) \rangle_{i\in\omega} = f(\sqcup_D \langle x_i \rangle_{i\in\omega}).$$

Let $K = \langle x_i \rangle_{i\in\omega}$ be an arbitrary chain in D. By the above condition there will be a natural number k such that

$$\forall i > k : x_i = x_k = \sqcup_D K.$$

So, we can consider the the finite chain $\{x_1, \ldots, x_k\}$. We have that $f(\sqcup_D\{x_1, \ldots, x_k\}) = f(x_k)$. Since f is monotonous, then

$$\forall i \in \{1, \ldots, k-1\} : f(x_i) \sqsubseteq_E f(x_{i+1})$$

that is f is a chain in E and thus

$$\sqcup_E\{f(x_1), \ldots, f(x_k)\} = f(x_k).$$

Furthermore, for all $i > k : f(x_i) = f(x_k)$ and thus:

$$\sqcup_E \langle f(x_i) \rangle_{i\in\omega} = f(x_k) = f(\sqcup_D K).$$

Ex. 3 La definition d'une fonction recursive peut etre donnée par la definition d'une relation binaire. Par example la fonction:

$$\begin{aligned} f(0) &= 1 \\ f(n) &= n * f(n-1) \end{aligned}$$

peut etre vue comme la relation $F \subseteq \mathbb{N} \times \mathbb{N}$ defini recursivement par

$$F = \{\langle 0, 1\rangle\} \cup \{\langle n, n*y\rangle \,|n > 0 \wedge \langle n-1, y\rangle \in F\}.$$

Montrer que l'ensemble

$$I = \{\left\langle n, \prod_{i=1}^{n} i \right\rangle |n > 0\}$$

est le point fixe de F, donc la solution de l'equation recursive pour F. Montrer aussi que: $\langle n, y\rangle \in I$ ssi $f(n) = y$.

Solution:

$F = \{\langle 0, 1\rangle\} \cup \{\langle n, n*y\rangle \,|n > 0 \wedge \langle n-1, y\rangle \in F\} =$

$= \{\langle 0, 1\rangle\} \cup \{\langle n, n*y\rangle \,|n > 0 \wedge \langle n-1, y\rangle \in \{\langle n, \prod_{i=1}^{n} i\rangle \,|n > 0\}\} =$

$= \{\langle 0, 1\rangle\} \cup \{\langle n, n*y\rangle \,|n > 0 \wedge \langle n-1, y\rangle = \langle n-1, \prod_{i=1}^{n-1} i\rangle\} =$

$= \{\langle 0, 1\rangle\} \cup \{\langle n, n * \prod_{i=1}^{n-1} i\rangle \,|n > 0\} =$

$= \{\langle 0, 1\rangle\} \cup \{\langle n, \prod_{i=1}^{n} i\rangle \,|n > 0\} = I$

We want to show also that: $\langle n, y\rangle \in I$ ssi $f(n) = y$.

This is easily proved by induction on n. It is true for $n = 0$. Assuming that it holds for a generic n let us show that it holds for $n + 1$.

By the inductive hypothesis we have that:

$$\langle n, y\rangle \in I$$

iff

$$y = \prod_{i=1}^{n} i = f(n)$$

On the other hand:

$$\langle n+1, y\rangle \in I$$

iff

$$y = \prod_{i=1}^{n+1} i = (n+1) * \prod_{i=1}^{n} i = (n+1) * f(n) = f(n+1).$$

Ex. 4 Soit le cpo $(D, \leq)$ et soit $([D \to D], \sqsubseteq)$ le cpo de fonctions continues telles que:

- $\forall f, g \in D : f \sqsubseteq g \Leftrightarrow \forall d \in D : f(d) \leq g(d)$

  Montrer que $([D \to D], \sqsubseteq)$ est un cpo.

Solution:

Soit $F = (D \to D) = \{f : D \to D | f\ continue\}$. Pour montrer que F est un cpo il faut montre que:

- $\sqsubseteq_F$ est transitive, reflexive et anti-symmetrique (trivial: pointwise definition).
- F est complet. Considerons une chaine $\langle f_i \rangle_i \subseteq F$. On construit f telle que $f(x) = lim_i f_i(x)$ dans D. Cette fonction $f : D \to D$ est continue (donc dans F) car, pour toute chaine $\langle x_k \rangle_k \subseteq D$ on a
$$f(lim_k x_k) = lim_i f(lim_k x_k) = lim_i lim_k f_i(x_k)$$
par construction de f et utilisant la continuité de f_i: $lim_k f(x_k) = lim_k lim_i f_i(x_k)$. On peut inverser l'ordre des limites grâce au lemme: $lim_i lim_k d_{ik} = lim_k lim_i d_{ik}$ pour toute "double chaine" $\langle d_{ik} \rangle_{i,k} \subseteq D$ telle que $d_{ik} \leq d_{i'k}$ si $i < i'$ et $d_{ik} \leq d_{ik'}$ si $k < k'$.
On montre facilement que le membre de gauche $lim_i lim_k d_{ik}$ est le l.u.b. de l'ensemble $\langle d_{ik} \rangle_{i,k}$ dans D. Par symétrie le membre de droite aussi.

Ex. 5 Soit les cpo's $(D_1, \leq_1)$ et $(D_2, \leq_2)$. Montrer que la structure $(D, \leq) = (D_1 + D_2, \leq)$, appelée *somme disjointe* et defini par:

- $D = (D_1 \times \{1\}) \cup (D_2 \times \{2\}) \cup \{\perp_D\}$
- $\forall \langle d, i \rangle, \langle d', j \rangle \in D : \langle d, i \rangle \leq \langle d', j \rangle \Leftrightarrow i = j \wedge d \leq_i d'$
- $\forall \langle d, i \rangle \in D : \perp_D \leq \langle d, i \rangle$

est un cpo.

Solution:

D est un ordre partiel: trivial car $x \leq_D y$ n'est a considérer que si $\{x, y\} \subseteq D_i$ (même composante) ou alors $x = \perp_D$.

D est complet: soit une chaine $\langle d_i \rangle_i \subseteq D$. Soit $d_i = \perp_D$ pour tout i et le $lim_i d_i = \perp_D$, soit $d_k \neq \perp_D$ à partir d'un certain k; dans ce cas ou se trouve dans une des composantes D_1 (ou D_2) pour $\langle d_i \rangle_{i \geq k} \subseteq D_1$ et $lim_i d_i$ existe dans D_1.
Remarque: $D_1 + D_2$ satisfait la propriété universelle duale du produit $D_1 \times D_2$.

Ex. 6 Soit les cpo's $(D_1, \leq_1)$ et $(D_2, \leq_2)$.

1. Montrer que la structure $(D, \leq) = (D_1 \oplus D_2, \leq)$, appelée *"smashed sum"* et defini par:
   - $D = ((D_1 \setminus \{\perp_{D_1}\}) \times \{1\}) \cup ((D_2 \setminus \{\perp_{D_2}\}) \times \{2\}) \cup \{\perp_D\}$
   - $\forall \langle d, i \rangle, \langle d', j \rangle \in D : \langle d, i \rangle \leq \langle d', j \rangle \Leftrightarrow i = j \wedge d \leq_i d'$
   - $\forall \langle d, i \rangle \in D : \perp_D \leq \langle d, i \rangle$

   est un cpo.

2. Utiliser la definition de "smashed sum" pour construire le domain $Alist$ des listes de type A comme solution de l'equation des domaines recursifs:
$$Alist = (Nil \oplus (A \times Alist))$$
   ou $Nil = \langle \perp, nil \rangle$ et $\perp \leq nil$.

Solution: 1

$D_1 \oplus D_2$ is a c.p.o. The proof is identical to the one in ex.5. Notice that $D_1 \oplus D_2$ satisfies the universal property for sums (dual to the cartesian product) with the restriction to strict monotonic functions $(f : f(\perp) = \perp)$.

Solution: 2

Alist satisfies equation

$$D = (Nil \oplus (A \times D))$$

A solution D of the above recursive equation can be constructed as the limit of the sequence of approximating domains $\langle D^i \rangle_i$ where each element of D^i will be contained in D and each D^i will be a sub-domain of D^{i+1} in the sense that the elements and the order in D^i will be preserved in D^{i+1}. Let's start with $D^0 = \{\perp\}$ since non trivial domains must be non-empty. The domain D^{i+1} will be obtained applying the recursive definition to D^i.

In our case, since $Alist^0 = \{\perp\}$ then $Alist^1 = (Nil \oplus (A \times Alist^0)) = Nil \oplus (A \times \{\perp\})$. This domain includes the empty list and all the lists of exactly one element over the domain A that is:

$$Alist^1 = \{nil, \perp\} \cup \{(a, \perp) | a \in A \wedge a \neq nil\}.$$

Analogously, $Alist^2 = (Nil \oplus (A \times Alist^1)) = (Nil \oplus (A \times (Nil \oplus (A \times \{\perp\}))))$. $Alist^2$ can be thought of as the union of distinct sets:

- $\{nil, \perp\}$
- $\{(a, \perp) | a \in A\}$: the set of partial lists of one element.
- $\{(a, nil) | a \in A \wedge a \neq nil\}$: the set of proper lists of one element.
- $\{(a_1, (a_2, \perp)) | a_1, a_2 \in A \wedge a, a' \neq nil\}$: the set of partial lists of two elements.

$Alist^2$ contains essentially all lists of length < 2.

Let $Alist^{fin} = \bigcup_{i=1}^{\infty} Alist^i$ where we put an ordering consistent with previous domains $Alist^i$, that is:

$$x \sqsubseteq_{Alist^{fin}} y \textbf{ iff } \exists j \geq 0 : x \sqsubseteq_{Alist^j} y.$$

$Alist^{fin}$ contains $\perp, nil$ and all proper lists of finite length, but it is *not* complete since it contains chains with no LUB such as those like:

$$\{\perp, (a_0, \perp), (a_0, a_1, \perp), \ldots\}.$$

It is possible to complete this domain adding the missing LUBs, that is adding the set of all possible infinite lists over the domain A. We will call this domain $Alist^{\infty}$ and it satisfies the above equation considering an isomorphism relation:

$$Alist^{\infty} \cong (Nil \oplus (A \times Alist^{\infty})).$$