Concurrency Semantics
Week 5

Course Notes 2005
EPFL – I&C

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April 27, 2005

4 Congruence & Reaction

4.1 Notation We often write $(\boldsymbol{\nu} ab)\,P$ instead of $(\boldsymbol{\nu} a)\,(\boldsymbol{\nu} b)\,P$.
We often omit trailing $.\mathbf{0}$ and abbreviate $a.\mathbf{0}$ by a.

4.2 Definition (Process Contexts) A *process context* $C[\cdot]$ is (precisely) defined by the following syntax:

$$\begin{array}{rcl} C[\cdot] & ::= & [\cdot] \;\big|\; \mu.C[\cdot] + M \;\big|\; M + \mu.C[\cdot] \\ & \big| & (\boldsymbol{\nu} a)\,C[\cdot] \;\big|\; C[\cdot]|P \;\big|\; P|C[\cdot] \end{array}$$

The *elementary process contexts* are

$$\begin{array}{ccc} (\boldsymbol{\nu} a)\,[\cdot] & \begin{array}{c} \mu.[\cdot] + M \\ [\cdot] \mid P \end{array} & \begin{array}{c} M + \mu.[\cdot] \\ P \mid [\cdot] \end{array} \end{array}$$

The expression $C[Q]$ denotes the result of filling the hole $[\cdot]$ of $C[\cdot]$ with process Q.

4.3 Definition (Process Congruence)
Let $\cong$ be an equivalence relation over $\mathcal{P}$.
Then $\cong$ is said to be a process congruence, if it is preserved by all elementary contexts; i.e., $P \cong Q$ implies all of the following:

$$\begin{array}{rclrcl} \mu.P + M & \cong & \mu.Q + M & P|R & \cong & Q|R \\ M + \mu.P & \cong & M + \mu.Q & R|P & \cong & R|Q \end{array}$$
$$(\boldsymbol{\nu} a)\,P \;\cong\; (\boldsymbol{\nu} a)\,Q$$

4.4 Proposition An arbitrary equivalence relation $\cong$ over processes $\mathcal{P}$ is a process congruence if and only if, for *all* contexts $C[\cdot]$, $P \cong Q$ implies $C[P] \cong C[Q]$.

4.5 Theorem
Bisimilarity $\sim$ is a process congruence.

4.6 Definition (Structural congruence)
Structural congruence, written $\equiv$, is the (smallest) process congruence over $\mathcal{P}$ determined by the following equations.

1. $=_\alpha$
2. commutative monoid laws for $(\mathcal{M}, +, \mathbf{0})$
3. commutative monoid laws for $(\mathcal{P}, \mid, \mathbf{0})$
4. $$\begin{array}{rcll} (\boldsymbol{\nu} a)\,(P \mid Q) & \equiv & P \mid (\boldsymbol{\nu} a)\,Q, & \text{if } a \notin \mathrm{fn}(P) \\ (\boldsymbol{\nu} ab)\,P & \equiv & (\boldsymbol{\nu} ba)\,P & \\ (\boldsymbol{\nu} a)\,\mathbf{0} & \equiv & \mathbf{0} & \end{array}$$

5. $\mathsf{A}\langle \vec{b} \rangle \equiv \{\vec{b}/\vec{a}\} M_A$, if $\mathsf{A}(\,\vec{a}\,) \stackrel{\mathrm{def}}{=} M_A$.

Note that n-ary summation is already implicitly defined modulo the commutative monoid laws.

4.7 Definition (Standard Form)
A process expression $(\boldsymbol{\nu}\vec{a})\ (\ M_1 | \cdots | M_n\ )$,
where each M_i is a non-empty sum,
is said to be in *standard form*.
If $n = 0$ then $M_1 | \cdots | M_n$ means **0**.
If $\vec{a}$ is empty then there is no restriction.

4.8 Theorem Every process is structurally congruent to some standard form.

4.9 Definition (Reaction Semantics)
The reaction relation $\rightarrow$ over $\mathcal{P}$
is generated precisely by the following rules:

$$\text{TAU: } \tau.P + M \ \rightarrow P$$

$$\text{REACT: } a.P + M \mid \overline{a}.Q + N \ \rightarrow P \mid Q$$

$$\text{PAR: } \frac{P \ \rightarrow P'}{P \mid Q \ \rightarrow P' \mid Q} \qquad \text{RES: } \frac{P \ \rightarrow P'}{(\boldsymbol{\nu}a)\, P \ \rightarrow (\boldsymbol{\nu}a)\, P'}$$

$$\text{STRUCT: } \frac{P \ \rightarrow P'}{Q \ \rightarrow Q'} \text{ IF } P \equiv Q \text{ AND } P' \equiv Q'$$

4.10 Proposition
If $P \ \xrightarrow{\mu} P'$ and $P \equiv Q$,
then there is Q' such that $Q \ \xrightarrow{\mu} Q'$ and $P' \equiv Q'$.

4.11 Corollary
$\equiv$ is a strong bisimulation.

4.12 Theorem
$P \ \rightarrow P'$ iff $P \ \xrightarrow{\tau}\equiv P'$.

4.13 Definition (Strong Simulation up to $\equiv$)
A binary relation S (on $\mathcal{P}$) is
a *(strong) simulation up to* $\equiv$
if, whenever $P\ S\ Q$, then if $P \ \xrightarrow{\mu} P'$ then
there is $Q' \in \mathcal{P}$ such that $Q \ \xrightarrow{\mu} Q'$ and $P' \equiv S \equiv Q'$.
S is a strong bisimulation up to $\equiv$
if its converse also has this property.

4.14 Proposition
If S is a (strong) bisimulation up to $\equiv$ and $P\ S\ Q$,
then $P \sim Q$.