Local confluence of the λ-calculus

We will show today that the λ-calculus is locally confluent (it is actually confluent but this is another story).

Local confluence

Show that if $M \in \Lambda$, $M \to_\beta M'$ and $M \to_\beta M''$ then there exists $N \in \Lambda$ such that $M' \to_\beta^\star N$ and $M'' \to_\beta^\star N$.

Proof:
We show this result by induction on M.

- If M is a variable x, then M is in β-normal form and the result trivially holds.
- If M is an abstraction $\lambda x.M_1$:

  Necessarily, we have $M' = \lambda x.M_1'$ and $M'' = \lambda x.M_1''$ with $M_1 \to_\beta M_1'$ and $M_1 \to_\beta M_1''$.

  By induction hypothesis, there exists N_1 such that $M_1' \to_\beta^\star N_1$ and $M_1'' \to_\beta^\star N_1$.

  So, we have $\lambda x.M_1' \to_\beta^\star \lambda x.N_1$ and $\lambda x.M_1'' \to_\beta^\star \lambda x.N_1$.

  Let N be $\lambda x.N_1$. Then $M' \to_\beta^\star N$ and $M'' \to_\beta^\star N$. The result holds.
- If M is an application $M_1 M_2$, there are several cases, depending on which part of M reduces to give M' and M'':
  - If $M' = M_1' M_2$ and $M'' = M_1'' M_2$ with $M_1 \to_\beta M_1'$ and $M_1 \to_\beta M_1''$. Then by induction hypothesis, there exists N_1 such that $M_1' \to_\beta^\star N_1$ and $M_1'' \to_\beta^\star N_1$.
    Let N be $N_1 M_2$. Then $M' \to_\beta^\star N$ and $M'' \to_\beta^\star N$.
  - If $M' = M_1 M_2'$ and $M'' = M_1 M_2''$ with $M_2 \to_\beta M_2'$ and $M_2 \to_\beta M_2''$. The same reasoning as above (but with M_2 instead of M_1) gives the result.
  - If $M' = M_1' M_2$ and $M'' = M_1 M_2''$ with $M_1 \to_\beta M_1'$ and $M_2 \to_\beta M_2''$.
    Let N be $M_1' M_2''$. Then $M' \to_\beta N$ and $M'' \to_\beta N$
  - If $M = (\lambda x.M_0) M_2$ and $M' = (\lambda x.M_0') M_2$ and $M'' = M_0 [{}^x/_{M_2}]$ with $M_0 \to_\beta M_0'$.
    Let N be $M_0' [{}^x/_{M_2}]$. We have that $M' \to_\beta N$. Moreover, by the third substitution lemma, we have that $M'' \to_\beta N$. So the result holds.
  - If $M = (\lambda x.M_0) M_2$ and $M' = (\lambda x.M_0) M_2'$ and $M'' = M_0 [{}^x/_{M_2}]$ with $M_2 \to_\beta M_2'$.
    Let N be $M_0 \left[{}^x/_{M_2'}\right]$. Then, we have $M' \to_\beta N$. By the first substitution lemma, we have that $M'' \to_\beta^\star N$ and so the result holds.

  In all possible cases, we have shown that there exists N such that $M' \to_\beta^\star N$ and $M'' \to_\beta^\star N$.

By induction principle, the result holds.

First Substitution Lemma

Show that if $M, N \in \Lambda$ and $N \to_\beta N'$ then $M\,[{}^{x}/_{N}] \to_\beta^\star M\,[{}^{x}/_{N'}]$.

Proof:
We show this by structural induction on the term M.

- If M is a variable z, there are two cases:
  - Either $z = x$:
    In this case, $M\,[{}^{x}/_{N}] = N \to_\beta N' = M\,[{}^{x}/_{N'}]$ and so the claim holds.
  - Or $z = y \neq x$
    In this case, $M\,[{}^{x}/_{N}] = y = M\,[{}^{x}/_{N'}]$ and we have $y \to_\beta^\star y$ and the claim also holds.
- If M is an abstraction $\lambda y.M'$ (we can choose y such that $y \neq x$ and $y \notin \mathrm{fv}(N) \cup \mathrm{fv}(N')$ by α-conversion):

  By induction hypothesis, $M'\,[{}^{x}/_{N}] \to_\beta^\star M'\,[{}^{x}/_{N'}]$ and so $\lambda y.(M'\,[{}^{x}/_{N}]) \to_\beta^\star \lambda y.(M'\,[{}^{x}/_{N'}])$.

  Now, $M\,[{}^{x}/_{N}] = \lambda y.(M'\,[{}^{x}/_{N}])$ because $x \neq y$ and $y \notin \mathrm{fv}(N) \cup \mathrm{fv}(N')$ and $M\,[{}^{x}/_{N'}] = \lambda y.(M'\,[{}^{x}/_{N'}])$ for the same reason.

  So, $M\,[{}^{x}/_{N}] \to_\beta^\star M\,[{}^{x}/_{N'}]$.
- If M is an application $M_1 M_2$:

  By induction hypothesis, $M_1\,[{}^{x}/_{N}] \to_\beta^\star M_1\,[{}^{x}/_{N'}]$ and $M_2\,[{}^{x}/_{N}] \to_\beta^\star M_2\,[{}^{x}/_{N'}]$, so $M\,[{}^{x}/_{N}] = M_1\,[{}^{x}/_{N}]\,M_2\,[{}^{x}/_{N}] \to_\beta^\star M_1\,[{}^{x}/_{N'}]\,M_2\,[{}^{x}/_{N'}] = M\,[{}^{x}/_{N'}]$.

We conclude that, by induction principle, the claim holds.

Second Substitution Lemma

Show that if $M, N, P \in \Lambda$, $x \notin \mathrm{fv}(P)$ and $x \neq y$ then $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right]$.

Proof:
By induction on M.

- If M is a variable z, there are three cases:
  - $z = x$
    Then, $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = N\,[{}^{y}/_{P}]$ and $M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = M\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = N\,[{}^{y}/_{P}]$.
  - $z = y$
    Then, $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = M\,[{}^{y}/_{P}] = P$ and $M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = P\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = P$ since $x \notin \mathrm{fv}(P)$.
  - $z \neq x$, $z \neq y$
    Then $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = M$ and $M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = M$.
- If M is an abstraction $\lambda z.M'$ with $z \neq x$, $z \neq y$, $z \notin \mathrm{fv}(N) \cup \mathrm{fv}(P)$ (possible by α-conversion):

  Then $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = \lambda z.(M'\,[{}^{x}/_{N}]\,[{}^{y}/_{P}])$.

  By induction hypothesis, $M'\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = M'\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right]$. Moreover, $M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right] = \lambda z.(M'\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right]$. So, $M\,[{}^{x}/_{N}]\,[{}^{y}/_{P}] = M\,[{}^{y}/_{P}]\,\left[{}^{x}/_{N[{}^{y}/_{P}]}\right]$.
- If M is an application $M_1 M_2$:

  We apply the induction hypothesis on M_1 and M_2 and we obtain the result.

By induction principle, the result holds.

Third Substitution Lemma

Show that if $M, N \in \Lambda$ and $M \to_\beta M'$ then $M\,[^x/_N] \to_\beta M'\,[^x/_N]$.

Proof:
By induction on M.

- If M is a variable z, then M is a β-normal form and the result trivially holds.
- If $M = \lambda z.M_1$ with $z \neq x$ and $z \notin \text{fv}(N)$:

  Then $M' = \lambda z.M_1'$ with $M_1 \to_\beta M_1'$. We have $M\,[^x/_N] = \lambda z.(M_1\,[^x/_N])$. By induction hypothesis, $M_1\,[^x/_N] \to_\beta M_1'\,[^x/_N]$ so $\lambda z.(M_1\,[^x/_N]) \to_\beta \lambda z.(M_1'\,[^x/_N])$. But $\lambda z.(M_1'\,[^x/_N]) = M'\,[^x/_N]$. So $M\,[^x/_N] \to_\beta M'\,[^x/_N]$.
- If $M = M_1 M_2$, there are several cases:
  - $M' = M_1' M_2$ with $M_1 \to M_1'$. By induction hypothesis $M_1\,[^x/_N] \to_\beta M_1'\,[^x/_N]$ so $M\,[^x/_N] \to_\beta M'\,[^x/_N]$.
  - $M' = M_1 M_2'$ with $M_2 \to M_2'$. Same reasoning as above.
  - $M = (\lambda z.M_0)M_2$ (with $z \neq x$ and $z \notin \text{fv}(N)$) and $M' = M_0\,[^z/_{M_2}]$.
    We have $M\,[^x/_N] = (\lambda z.(M_0\,[^x/_N]))M_2\,[^x/_N]$ which β-reduces to $M_0\,[^x/_N]\,\left[^z/_{M_2[^x/_N]}\right]$. By the second substitution lemma, we have that $M'\,[^x/_N] = M_0\,[^z/_{M_2}]\,[^x/_N] = M_0\,[^x/_N]\,\left[^z/_{M_2[^x/_N]}\right]$. So, we have $M\,[^x/_N] \to_\beta M'\,[^x/_N]$.

By induction principle, the result holds.