Local confluence of the λ-calculus

We will show today that the λ-calculus is locally confluent (it is actually confluent but this is another story).

Local confluence

Show that if $M \in \Lambda$, $M \to_\beta M'$ and $M \to_\beta M''$ then there exists $N \in \Lambda$ such that $M' \to_\beta^\star N$ and $M'' \to_\beta^\star N$.

First Substitution Lemma

Show that if $M, N \in \Lambda$ and $N \to_\beta N'$ then $M\,[{}^x/_N] \to_\beta^\star M\,[{}^x/_{N'}]$.

Second Substitution Lemma

Show that if $M, N, P \in \Lambda$, $x \notin \mathrm{fv}(P)$ and $x \neq y$ then $M\,[{}^x/_N]\,[{}^y/_P] = M\,[{}^y/_P]\,\left[{}^x/_{N[{}^y/_P]}\right]$.

Third Substitution Lemma

Show that if $M, N \in \Lambda$ and $M \to_\beta M'$ then $M\,[{}^x/_N] \to_\beta M'\,[{}^x/_N]$.