Concurrency: Theory, Languages and Programming

– Functional Programming and Lambda Calculus –

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Part I: Functional Programming

- A pure functional program consists of data, functions, and an expression which describes a result.
- Missing: variables, assignment, side-effects.
- A processor of a functional program is essentially a calculator.

xample: (transcript of a session with *siris*, the Scala nterpreter)

```
/home/odersky/tmp> siris
> def gcd (a: Int, b: Int): Int = if (b == 0) a else gcd (b, a % b)
'def gcd'
> gcd (8, 10)
2
> val x = gcd (15, 70)
val x : int = 5
> val y = gcd(x, x)
val y : int = 5
```


Why Study Functional Programming?

- FP is programming in its simplest form $\Rightarrow$ easier to understand thoroughly than more complex variants.
- FP has powerful composition constructs.
- In FP, one the *value* of an expression matters since side effects are impossible. (this property is called *referential transparency*).
- Referential transparency gives a rich set of laws to transform programs.
- FP has a well-established theoretical basis in Lambda Calculus and Denotational Semantics.

Square Roots by Newton's Method

ompute the square root of a given number x as a limit of the equence y_i given by:

$$\begin{aligned} y_0 &= 1 \\ y_{i+1} &= (y_i + x/y_i)/2 \end{aligned}$$

The $i \to i+1$ step is encoded in the function *improve*:

```
> def improve (guess: Double, x: Double) = (guess + x / guess) / 2
def improve : (guess : double,x : double)double
> val y0 = 1.0
val y0 : double = 1.0
> val y1 = improve (y0, 2.0)
val y1 : double = 1.5
> val y2 = improve (y1, 2.0)
val y2 : double = 1.4166666666666665
> val y3 = improve (y2, 2.0)
val y3 : double = 1.4142156862745097
```


We have to stop the iteration when the result is good enough:

```
> def abs(x: Double): Double = if (x ≥ 0) x else −x
def abs : (x : double)double
> def goodEnough (guess: Double, x: Double): Boolean =
|   abs((guess ∗ guess) − x) < 0.001
def goodEnough : (guess : double,x : double)boolean
> def sqrtIter(guess: Double, x: Double): Double =
|   if (goodEnough(guess, x)) guess else sqrtIter (improve(guess, x), x)
def sqrtIter : (guess : double,x : double)double
> def sqrt(x: Double): Double = sqrtIter(1.0, x)
def sqrt : (x : double)double
> sqrt (2.0)
1.4142156862745097
```


Language Elements Seen So Far

- Function Definitions:

  def Ident Parameters [*':' ResultType*] *"=" Expression*

- Value definitions:

  val Ident "=" Expression

- Function application: *Ident'('* $Expr_1$*, ...,* $Expr_2$ *')'*
- Tuples: (*Expr*$_1$*, ...,* $Expr_n$)
- Numbers, operators: as in Java
- If-then-else: as in Java, but as an expression.
- Types: as in Java, but written upper case.

Nested Functions

functions are used only internally by some other function we an avoid "name-space pollution" by nesting. E.g:

```
def sqrt (x) = {
  def improve (guess, x) = (guess + x / guess) / 2
  def goodEnough (guess, x) = abs ((guess * guess) − x) < 0.001f
  def sqrtIter (guess, x) =
    if (goodEnough (guess, x)) guess
    else sqrtIter (improve (guess, x), x)
  sqrtIter (1.0, x)
}
```

he visibility of an identifier extends from its own definition to the nd of the enclosing block, including any nested definitions.

Exercise:

- The *goodEnough* function tests the absolute difference between the input parameter and the square of the guess
- This is not very accurate for square roots of very small numbers and might lead to divergence for very large numbers (why?).
- Design a different *sqrtIter* function which stops if the *chan* from one iteration to the next is a small fraction of the guess. E.g.

$$abs((x_{i+1} - x_i)/x_i) < 0.001$$

Complete:

```
def sqrtIter(guess: Double, x: Double) = ?
```


Semantics of Function Application

- One simple rule: A function application *f* (*A*) is evaluated by
  - replacing the application with the function's body where
  - actual parameters *A* replace formal parameters of *f*.
- This can be formalised as a *rewriting of the program itself*:

  def *f* (*x*) = *B* ; ... *f* (*A*) → **def** *f* (*x*) = *B* ; ... [*A*/*x*] *B*

- Here, [*A*/*x*] *B* stands for *B* with all occurrences of *x* replaced by *A*.
- [*A*/*x*] *B* is called a *substitution*.

Rewriting Example:

Consider *gcd*:

```
def gcd(a: Int, b: Int) = if (b == 0) a else gcd (b, a % b)
```

Then *gcd* (*14, 21*) evaluates as follows:

```
      gcd (14, 21)
→     if (21 == 0) 14 else gcd (21, 14 % 21)
→     gcd (21, 14)
→     if (14 == 0) 21 else gcd (14, 21 % 14)
→→    gcd (14, 7)
→     if (7 == 0) 14 else gcd (7, 14 % 7)
→→    gcd (7, 0)
→     if (0 == 0) 7 else gcd (0, 7 % 0)
→     7
```

Another rewriting example:

onsider *factorial*:

def *factorial* $(n : Int) =$ ***if*** $(n == 0)\ 1$ ***else*** $n * factorial\ (n - 1)$

hen *factorial*(5) rewrites as follows:

$$
\begin{aligned}
& factorial\ (5) \\
\rightarrow\ & \textbf{if}\ (5 == 0)\ 1\ \textbf{else}\ 5 * factorial\ (5 - 1) \\
\rightarrow\ & 5 * factorial\ (5 - 1) \\
\rightarrow\ & 5 * factorial\ (4) \\
\rightarrow ...\rightarrow\ & 5 * (4 * factorial\ (3)) \\
\rightarrow ...\rightarrow\ & 5 * (4 * (3 * factorial\ (2))) \\
\rightarrow ...\rightarrow\ & 5 * (4 * (3 * (2 * factorial\ (1)))) \\
\rightarrow ...\rightarrow\ & 5 * (4 * (3 * (2 * (1 * factorial\ (0))))) \\
\rightarrow ...\rightarrow\ & 5 * (4 * (3 * (2 * (1 * 1)))) \\
\rightarrow ...\rightarrow\ & 120
\end{aligned}
$$

Question:

What differences are there between the two rewrite sequence

Tail Recursion

- Implementation note: If a function calls itself as its last action, the function's stack frame can be re-used. This is called "tail recursion".
- $\Rightarrow$ Tail-recursive functions are iterative processes.
- More generally, if the last action of a function is a call to another (possible the same) function, only a single stack frame is needed for both functions. Such calls are called "tail calls".

xercise: Design a tail-recursive version of *factorial*.

First-Class Functions

- Most functional languages treat functions as "first-class values".
- That is, like any other value, a function may be passed as parameter or returned as a result.
- This provides a flexible mechanism for program composition.
- Functions which take other functions as parameters or return them as results are called "higher-order" functions..

Example

- Sum integers between *a* and *b*:

```
def sumInts (a: Int, b: Int): Double =
  if (a > b) 0.0 else a + sumInts (a + 1, b);
```

- Sum cubes of all integers between *a* and *b*:

```
def cube (a: Int) = a * a * a;
def sumCubes (a: Int, b: Int): Double =
  if (a > b) 0.0 else cube (a) + sumCubes (a + 1, b);
```

- Sum reciprocals between *a* and *b*

```
def sumReciprocals (a: Int, b: Int): Double =
  if (a > b) 0 else 1.0 / a + sumReciprocals (a + 1, b);
```

- These are all special cases of $\sum_a^b f(n)$ for different values of f.

Summation with a higher-order functio

- Can we factor out the common pattern?
- Define:

```
def sum(f: (Int)Double, a: Int, b: Int): Double =
  if (a > b) 0.0 else f(a) + sum(f, a + 1, b);
```

- Then we can write:

```
def sumInts(a: Int, b: Int) = sum(id, a, b)
def sumCubes(a: Int, b: Int) = sum(cube, a, b)
def sumReciprocals(a: Int, b: Int) = sum(reciprocal, a, b)
```

where

```
def id(x: Int) = x
def cube(x: Int) = x * x * x
def reciprocal(x: Int) = 1.0 / x
```


Anonymous functions

- Parameterisation by functions tends to create many small functions.
- Sometimes it is cumbersome to have to define the functions using ***def***.
- A shorter notation makes use of *anonymous functions*, defined as follows:

  $(x_1 : T_1, ..., x_n : T_n \Rightarrow E)$ defines a function which maps its parameters $x_1, ..., x_n$ to the result of the expression E (where E may refer to $x_1, ..., x_n$).
- The parameter types T_i may be omitted if they can be reconstructed “from the context”.

- Anonymous functions are not essential in Scala; an anonymous function $(x_1, ..., x_n \Rightarrow E)$ can always be expressed using a ***def*** as follows:

  $\{$ ***def*** $f(x_1 : T_1, ..., x_n : T_n) = E \; ; \; f \}$

  where f is fresh name which is used nowhere else in the program.
- We also say, anonymous functions are “syntactic sugar”.

ummation with Anonymous Functions

ow we can write shorter:

```
def sumInts(a: Int, b: Int) = sum((x ⇒ x), a, b)
def sumCubes(a: Int, b: Int) = sum((x ⇒ x * x * x), a, b)
def sumReciprocals(a: Int, b: Int) = sum((x ⇒ 1.0 / x), a, b)
```

an we do even better?

int: *a, b* appears everywhere and does not seem to take part in

teresting combinations. Can we get rid of it?

Currying

Let's rewrite *sum* as follows.

```
def sum(f: (Int)Double) = {
  def sumFun (a: Int, b: Int): Double =
    if (a > b) 0.0
    else f(a) + sumFun(a + 1, b);
  sumFun
}
```

- *sum* is now a function which returns another function;
- Namely, the specialized summing function which applies t
  f function and sums up the results.

hen we can define:

```
val sumInts = sum (x ⇒ x)
val sumCubes = sum (x ⇒ x * x * x)
val sumReciprocals = sum (x ⇒ 1.0 / x)
```

unction values can be applied like other functions:

```
sumReciprocals (1, 1000)
```


Curried Application

How are function-returning functions applied?
Example:

```
> sum (cube) (1, 10)
3025
```

- *sum* (*cube*) applies *sum* to *cube* and returns the "cube-summing function" (Hence, *sum* (*cube*) is equivalen to *sumCubes*).
- This function is then applied to the pair (*1, 10*).
- Hence, function application associates to the left:

```
sum (cube) (1, 10)  ==  (sum (cube)) (1, 10)
                    ==  val sc = sum (cube) ; sc (1, 10)
```

Curried Definition

- The style of function-returning functions is so useful in FP, that we have special syntax for it.
- For instance, the next definition of *sum* is equivalent to the previous one, but shorter:

```
def sum (f: (Int, Int)Double) (a; Int, b: Int): Double = {
  if (a > b) 0.0
  else f(a) + sum(f)(a + 1, b)
}
```


Generally, a curried function definition

def *f* (*args*$_1$) ... (*args*$_n$) = *E*

where $n > 1$ expands to

def *f* (*args*$_1$) ... (*args*$_{n-1}$) = (**def** *g* (*args*$_n$) = *E* ; *g*)

where *g* is a fresh identifier. Or, shorter:

def *f* (*args*$_1$) ... (*args*$_{n-1}$) = (*args*$_n$ $\Rightarrow$ *E*)

Performing this step n times yields that

def *f* (*args*$_1$) ... (*args*$_{n-1}$) (*args*$_n$) = *E*

is equivalent to

def *f* = (*args*$_1$ $\Rightarrow$ (*args*$_2$ $\Rightarrow$ ... (*args*$_n$ $\Rightarrow$ *E*) ...))

- Again, parentheses around single-name formal parameters may be dropped.
- This style of function definition and application is called *currying* after its promoter, Haskell B. Curry.
- Actually, the idea goes back further to Frege and Schönfinkel, but the name “curried” caught on (maybe because “schönfinkeled” does not sound so well.)

Exercises:

1. The *sum* function uses a linear recursion. Can you write a tail-recursive one by filling in the ??’s?

```
def sum f (a: Int, b: Int): Double = {
  def iter (a: Int, result: Double): Double = {
    if (??) ??
    else iter (??, ??)
  }
  iter (??, ??)
}
```

. Write a function *product* that computes the product of the alues of functions at points over a given range.

. Write *factorial* in terms of *product*.

. Can you write an even more general function which eneralizes both *sum* and *product*?

Part II: Lambda Calculus

- Lambda Calculus is a foundation for functional programs.
- It's an operational semantics, based on term rewriting.
- Lambda Calculus was developed by Alonzo Church in the 1930's and 40's as a theory of computable functions.
- Lambda calculus is as powerful as Turing machines. That is, every Turing machine can be expressed as a function i the calculus and vice versa
- Church Hypothesis: Every computable algorithm can be expressed by a function in Lambda calculus.

Pure Lambda Calculus

- Pure Lambda calculus expresses only functions and function applications.
- Three term forms:

Names	$x, y, z \in \mathcal{N}$			
Terms	D, E, F	$::=$	x	names
		$\mid$	$\lambda x.E$	abstractions
		$\mid$	$D\ E$	applications

- We generally omit parentheses around single-name function arguments.
- Function-application is left-associative.
- The scope of a name extends as far to the right as possible.

- Example:

$$\lambda f.\lambda x.f\ E\ x \quad \equiv \quad (\lambda f.(\lambda x.((f\ (E))\ (x)))).$$

- Often, one uses the term *variable* instead of *name*.

Evaluation of Lambda Terms

valuation of lambda terms is by the β-reduction rule.

$$\beta: \qquad (\lambda x.D)E \;\rightarrow\; [E/x]\,D$$

$E/x]$ is substitution, which will be explained in detail later.

xample:

$$\begin{array}{lcl}
(\lambda x.x)(\lambda y.y) & \rightarrow & \lambda y.y \\
(\lambda f.\lambda x.f\ (f\ x))(\lambda y.y)z & \rightarrow & (\lambda x.(\lambda y.y)(\lambda y.y)x)z \\
& \rightarrow & (\lambda y.y)((\lambda y.y)z) \\
& \rightarrow & (\lambda y.y)z \\
& \rightarrow & z
\end{array}$$

Term Equivalence

Question: Are these terms equivalent?

$$\lambda x.x \qquad \text{and} \qquad \lambda y.y$$

What about

$$\lambda x.y \qquad \text{and} \qquad \lambda x.z$$

?

Need to distinguish between *bound* and *free* names.

Free And Bound Names

efinition The free names $\mathrm{fn}(E)$ of a term E are those names hich occur in E at a position where they are not in the scope of definition in the same term.

ormally, $\mathrm{fn}(E)$ is defined as follows.

$$\begin{array}{lcl}
\mathrm{fn}(x) & = & \{x\} \\
\mathrm{fn}(\lambda x.E) & = & \mathrm{fn}(E)\backslash\{x\} \\
\mathrm{fn}(F\ E) & = & \mathrm{fn}(F) \cup \mathrm{fn}(E).
\end{array}$$

ll names which occur in a term E and which are not free in E re called *bound*.

term without any free variables is called *closed*.

Renaming

- The spelling of bound names is not significant.
- We regard terms D and E which are convertible by renaming of bound names as equivalent, and write $D \equiv E$
- This is expressed formally by the following α-renaming rul

$$\alpha: \qquad \lambda x.E \;\equiv\; \lambda y.[y/x]E \qquad (y \notin \mathrm{fn}(E))$$

Formally, $\equiv$ is the smallest *congruence* which contains the equality of rule α.

Substitutions

- We now have the means to define substitution formally:

$$\begin{array}{lcll} [D/x]\ x & = & D & \\ [D/x]\ y & = & y & (x \neq y) \\ [D/x]\ \lambda x.E & = & \lambda x.E & \\ [D/x]\ \lambda y.E & = & \lambda y.[D/x]E & (x \neq y, y \notin \mathrm{fn}(D)) \\ [D/x]\ (F\ E) & = & ([D/x]F)\ ([D/x]E) & \end{array}$$

- Substitution affects only the free names of a term, not the bound ones.

Avoiding Name Capture

- We have to be careful that we do not bind free names of a substituted expression (this is called *name capture*).
- For instance,

$$[y/x]\lambda y.x \not\equiv \lambda y.y \qquad \text{!!!}$$

- We have to α-rename $\lambda y.x$ first before applying the substitution:

$$\begin{array}{lcll} [y/x]\lambda y.x & \equiv & [y/x]\lambda z.x & \text{by } \alpha \\ & \equiv & \lambda z.y & \end{array}$$

- In the following, we will always assume that terms are renamed automatically so as to make all substitutions well-defined.

Normal Forms

efinition: We write $\twoheadrightarrow$ for reduction in an arbitrary number of teps. Formally:

$$E \twoheadrightarrow E' \qquad \text{iff} \qquad \exists n \geq 0. E \equiv E_0 \rightarrow \ldots \rightarrow E_n \equiv E'$$

efinition: A *normal form* is a term which cannot be reduced urther.

xercise: Define:

$$\begin{array}{lcl} S & \stackrel{\mathrm{def}}{\equiv} & \lambda f.\lambda g.\lambda x.fx(gx) \\ K & \stackrel{\mathrm{def}}{\equiv} & \lambda x.\lambda y.x \end{array}$$

an SKK be reduced to a normal form?

Combinators

- Lambda calculus gives one the possibility to define new functions using λ abstractions.
- **Question:** Is that really necessary for expressiveness, or could one also do with a fixed set of functions?
- **Answer:** (by Haskell Curry) Every closed λ-definable functi can be expressed as some combination of the *combinato* S and K.

Combinator Implementation Technique

- This insight has influenced the implementation of one functional language (Miranda).
- The Miranda compiler translates a source program to a combination of a handful of combinators (S, K, and a few others for "optimizations").
- A Miranda runtime system then only has to implement the handful of combinators.
- Very elegant, but "slow as continental drift".

Confluence

If a term had more than one normal form, we'd have to worry about an implementation finding "the right one".
The following important theorem shows that this case cannot arise.

Theorem: (Church-Rosser) Reduction in λ-calculus is *confluent*: If $E \twoheadrightarrow E_1$ and $E \twoheadrightarrow E_2$, then there exists a term E_3 such that $E_1 \twoheadrightarrow E_3$ and $E_2 \twoheadrightarrow E_3$.
Proof: Not easy.

Corollary: Every term can be reduced to at most one norma form.
Proof: Your turn.

Terms Without Normal Forms

- There are terms which do not have a normal form.
- Example: Let

$$\Omega \stackrel{\text{def}}{\equiv} (\lambda x.(xx))(\lambda x.(xx))$$

Then

$$\begin{aligned} \Omega &\rightarrow (\lambda x.(xx))(\lambda x.(xx)) \\ &\rightarrow (\lambda x.(xx))(\lambda x.(xx)) \\ &\rightarrow \ldots \end{aligned}$$

- Terms which cannot be reduced to a normal form are called *divergent*.

Evaluation Strategies

The existence of terms without normal forms raises the question of *evaluation strategies*.

For instance, let $I \stackrel{\text{def}}{\equiv} \lambda x.x$ and consider:

$$\begin{aligned} & (\lambda x.I)\ \Omega \\ \rightarrow\ & I \end{aligned}$$

in a single step. But one could also reduce:

$$\begin{aligned} & (\lambda x.I)\ \Omega \\ \rightarrow\ & (\lambda x.I)\ \Omega \\ \rightarrow\ & \ldots \end{aligned}$$

by always doing the $\Omega \rightarrow \Omega$ reduction.

Complete Evaluation Strategies

n evaluation strategy is a decision procedure which tells us hich rewrite step to choose, given a term where several eductions are possible.

uestion 1: Is there a *complete* evaluation strategy, in the llowing sense:

> Whenever a term has a normal form, the reduction using the strategy will end in that normal form.

Weak Head Normal Forms

In practice, we are not so much interested in normal forms; or in terms which are not further reducible "at the top level".
That is, reduction would stop at a term of the form $\lambda x.E$ even E was still reducible.
These terms are called *weak head normal forms* or *values*.
They are characterized by the following grammar.

$$\text{Values} \quad V \quad ::= \quad x \mid \lambda x.E$$

We now reformulate our question as follows:

Question 2: Is there a (weakly) complete evaluation strategy, in the following sense:

> Whenever a term can be reduced to a value, the reduction using the strategy will end in that value.

recise Definition of Evaluation Strategy

- How can we define evaluation strategies formally?
- **Idea:** Use *reduction contexts*.

efinition: A *context* C is a term where exactly one subterm is eplaced by a "hole", written $[\,]$.
$[E]$ denotes the term which results if the hole of context C is lled with term E.

- Examples of contexts:

$$[\,] \qquad \lambda x.\lambda y.[\,] \qquad \lambda x.f\ [\,]$$

- Previously, we have admitted reduction anywhere in a ter without explicitly saying so.
- Let's formalize this:

Definition: A term E *reduces at top-level* to a term E', if E a E' are the left- and right-hand sides of an instance of rule β. W write in this case: $E \to_\beta E'$.

Definition: A term E *reduces* to a term E', written $E \to E'$ if there exists a context C and terms D, D' such that

$$\begin{array}{lcl} E & \equiv & C[D] \\ E' & \equiv & C[D'] \\ D & \to_\beta & D' \end{array}$$

- So much for general reduction.
- Now, to define an evaluation strategy, we *restrict* the possible set of contexts in the definition of $\rightarrow$.
- The restriction can be expressed by giving a *grammar* which describes permissible contexts.
- Such contexts are called *reduction contexts* and we let the letter R range over them

Call-By-Name

Definition: The *call-by-name* strategy is given by the followir grammar for reduction-contexts:

$$R \quad ::= \quad [\,] \mid R\ E$$

Definition: A term E *reduces* to a term E' using the call-by-name strategy, written $E \rightarrow_{\mathsf{cbn}} E'$ if there exists a reduction context R and terms D, D' such that

$$\begin{array}{lcl} E & \equiv & R[D] \\ E' & \equiv & R[D'] \\ D & \rightarrow_\beta & D' \end{array}$$

Deterministic Reduction Strategies

efinition: A reduction strategy is *deterministic* if for any term t most one reduction step is possible.

roposition: The call-by-name strategy $\rightarrow_{\mathsf{cbn}}$ is deterministic.

roof: There is only one way a term can be split into a reduc-

on context *R* and a subterm which is reducible at top-level.

Exercise: Reduce the term $K\ I\ \Omega$ with the call-by-name strategy, where

$$\begin{array}{lcl} K & \stackrel{\mathrm{def}}{\equiv} & \lambda x.\lambda y.x \\ I & \stackrel{\mathrm{def}}{\equiv} & \lambda x.x \\ \Omega & \stackrel{\mathrm{def}}{\equiv} & (\lambda x.(xx))(\lambda x.(xx)) \end{array}$$

Theorem: (Standardization) Call-by-name reduction is weak complete: Whenever $E \twoheadrightarrow V$ then $E \twoheadrightarrow_{\mathsf{cbn}} V'$.

Proof: hard.

Normal Order Reduction

uestion:

- Modify call-by-name reduction to *normal-order reduction*, which always reduces a term to a normal form, if it has one.
- Which changes to the definition of reduction contexts R are necessary?

- In practice, call-by-name is rarely used since it leads to duplicate evaluations of arguments. Example:

$$\begin{aligned} & (\lambda f.f(fy))((\lambda x.x)(\lambda x.x)) \\ \rightarrow\ & (\lambda x.x)(\lambda x.x)((\lambda x.x)(\lambda x.x)y) \\ \rightarrow\ & (\lambda x.x)((\lambda x.x)(\lambda x.x)y) \\ \rightarrow\ & (\lambda x.x)((\lambda x.x)y) \\ \rightarrow\ & (\lambda x.x)y \\ \rightarrow\ & y \end{aligned}$$

- Note that the argument $(\lambda x.x)(\lambda x.x)$ is evaluated twice.

- A shorter reduction can often be achieved by evaluating function arguments before they are passed. In our example:

$$\begin{aligned} & (\lambda f.f(fy))((\lambda x.x)(\lambda x.x)) \\ \rightarrow\ & (\lambda f.f(fy))(\lambda x.x) \\ \rightarrow\ & (\lambda x.x)((\lambda x.x)y) \\ \rightarrow\ & (\lambda x.x)y \\ \rightarrow\ & y \end{aligned}$$

Call-By-Value

- The *call-by-value* strategy evaluates function arguments before applying the function.
- It is often more efficient than the call-by-name strategy. However:

Proposition: The call-by-value strategy is not (weakly) complete.

- **Question:** Name a term which can be reduced to a value following the call-by-name strategy, but not following the call-by-value strategy.
- Hence we have a dilemma: One strategy is in practice too inefficient, the other is incomplete.

irst Solution: Call-By-Need Evaluation

- **Idea:** Rather than re-evaluating arguments repeatedly, save the result of the first evaluation and use that for subsequent evaluations.
- This technique is called *memoization*.
- It is used in implementations of *lazy* functional languages such as Miranda or Haskell.
- A formalization of call-by-need is possible, but beyond the scope of this course. See

  A Call-by-Need Lambda Calculus, Zena Ariola, Matthias Felleisen, John Maraist, Martin Odersky and Philip Wadler. *Proc. ACM Symposium on Principles of Programming Languages*, 1995.

Second Solution: Call-By-Value Calcul

- Rather than tweaking the evaluation strategy to be comple with respect to a given calculus, we can also change the calculus so that a given evaluation strategy becomes complete with respect to it.
- This has been done by Gordon Plotkin, in the *call-by-valu* lambda calculus.
- The *terms* and *values* of this calculus are defined as befo A more concise re-formulation is:

$$\begin{array}{lllll} \text{Terms} & D, E, F & ::= & V \mid D\ E \\ \text{Values} & V, W & ::= & x \mid \lambda x.E \end{array}$$

- As reduction rule, we have:

$$\beta_{\mathbf{V}}: \qquad (\lambda x.D)V \ \rightarrow \ [V/x]\ D$$

- As reduction contexts, we have:

$$R_V \ ::= \ [\,] \mid R_V\ E \mid V\ R_V$$

- Let $\rightarrow_V$ be general reduction of terms with the β_V rule, an let $\rightarrow_{\mathsf{cbv}}$ be β_V reduction only at the holes of call-by-value reduction contexts R_V. Then we have:

Theorem: (Plotkin) $\rightarrow_V$ reduction is confluent.

Theorem: (Plotkin) $\rightarrow_{\mathsf{cbv}}$ is weakly complete with respect $\rightarrow_V$.