# Concurrency: Theory, Languages and Programming

# – Equivalences for CCS –

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## Bisimulation on CCS

- check out Session 4, again
- add 1+1 . . .

- $\beta.P + \beta.P + M \sim \beta.P + M$
- $(\boldsymbol{\nu} a)\, a.P \sim \mathbf{0}$
- $(\boldsymbol{\nu} a)\, \overline{a}.P \sim \mathbf{0}$
- $(\boldsymbol{\nu} c)\, (\ a.c.P \mid b.\overline{c}.Q\ ) \sim (\boldsymbol{\nu} c)\, (\ a.c.Q \mid b.\overline{c}.P\ )$
- ...
- Why *algebraic* ?



# "Algebraic" Properties (II)

- $a \mid b \sim a.b + b.a$
- For all $P \in \mathcal{P}$, $P \sim \sum\{\ \beta.Q \mid P \xrightarrow{\beta} Q\ \}$.
- For all $n \geq 0$ and $P_1, \ldots, P_n \in \mathcal{P}$:

$$P_1|\cdots|P_n \sim \left\{ \begin{array}{l} \sum\{\ \beta.(\ P_1|\cdots|P_i'|\cdots|P_n\ ) \\ \qquad \mid 1{\leq}i{\leq}n, P_i \xrightarrow{\beta} P_i'\ \} \\ + \\ \sum\{\ \tau.(\ P_1|\cdots|P_i'|\cdots|P_j'|\cdots|P_n\ ) \\ \qquad \mid 1{\leq}i{<}j{\leq}n, P_i \xrightarrow{\lambda} P_i', P_j \xrightarrow{\overline{\lambda}} P_j'\ \} \end{array} \right.$$

For all $n \geq 0$, $P_1, \ldots, P_n \in \mathcal{P}$, and $\vec{a}$:

$$(\boldsymbol{\nu}\vec{a})\,(\,P_1|\cdots|P_n\,) \sim \left\{ \begin{array}{l} \sum\{\ \beta.(\boldsymbol{\nu}\vec{a})\,(\,P_1|\cdots|P_i'|\cdots|P_n\,) \\ \qquad \mid 1{\leq}i{\leq}n, P_i \xrightarrow{\beta} P_i',\ \text{and}\ \beta, \overline{\beta} \notin \vec{a}\ \} \\ + \\ \sum\{\ \tau.(\boldsymbol{\nu}\vec{a})\,(\,P_1|\cdots|P_i'|\cdots|P_j'|\cdots|P_n\,) \\ \qquad \mid 1{\leq}i{<}j{\leq}n, P_i \xrightarrow{\lambda} P_i', P_j \xrightarrow{\overline{\lambda}} P_j'\ \} \end{array} \right.$$

***Expansion Law !*** (also called: *Interleaving*)
Compare to the notions of *standard forms* in Milner's book: every process term can be transformed into a form that matches the left-hand side of the above equation.



# Process Contexts

**Definition:**A **process context** $C[\cdot]$ is (precisely) defined by the following syntax:

$$\begin{array}{rcl} C[\cdot] & ::= & [\cdot] \mid \alpha.C[\cdot] + M \mid M + \alpha.C[\cdot] \\ & \mid & (\boldsymbol{\nu}a)\,C[\cdot] \mid C[\cdot]|P \mid P|C[\cdot] \end{array}$$

The **elementary contexts** are
$\alpha.[\cdot] + M$, $M + \alpha.[\cdot]$, $(\boldsymbol{\nu}a)\,[\cdot]$, $[\cdot]|P$, $P|[\cdot]$.

$C[Q]$ denotes the result of filling the hole $[\cdot]$ of $C[\cdot]$ with process $Q$.

**Definition:**(Process congruence)
Let $\cong$ be an *equivalence relation* over $\mathcal{P}$.

Then $\cong$ is said to be a ***process congruence***, if
for *all* contexts $C[\cdot]$,
$P \cong Q$ implies $C[P] \cong C[Q]$.



# Process congruence (II)

**Proposition:**
An arbitrary equivalence relation $\cong$ is a process congruence if, and only if, it is preserved by all *elementary contexts*;
i.e., if $P \cong Q$, then

$$\begin{array}{rclcrcl} \alpha.P + M & \cong & \alpha.Q + M & \qquad & P|R & \cong & Q|R \\ M + \alpha.P & \cong & M + \alpha.Q & & R|P & \cong & R|Q \end{array}$$

$$(\boldsymbol{\nu} a)\, P \;\cong\; (\boldsymbol{\nu} a)\, Q \;\; .$$

**Note:**
For proving that an equivalence relation is a congruence, the elementary contexts suffice.

**Proposition:**
Bisimilarity is a process congruence, i.e., . . .



# Towards Observation Equivalence

Let us assume that our LTSs may dispose of a single distinguished *internal action* symbol, say: $\tau$, as is the case for our language of concurrent process expressions. Then:

**"Different internal behavior" should "not count" !**

**Definition:**( observations / weak actions )

1. $\Rightarrow \stackrel{\text{def}}{=} \stackrel{\tau}{\longrightarrow}^*$
2. $\stackrel{\lambda}{\Longrightarrow} \stackrel{\text{def}}{=} \Rightarrow \stackrel{\lambda}{\longrightarrow} \Rightarrow$

**Definition:**
$\mathcal{S}$ is a weak simulation **iff**, whenever $P \; \mathcal{S} \; Q$,

- if $P \xrightarrow{\tau} P'$ then there is $Q' \in \mathcal{P}$ such that $Q \Rightarrow Q'$ and $P' \; \mathcal{S} \; Q'$.
- if $P \xrightarrow{\lambda} P'$ then there is $Q' \in \mathcal{P}$ such that $Q \stackrel{\lambda}{\Longrightarrow} Q'$ and $P' \; \mathcal{S} \; Q'$.

$q$ **weakly simulates** $p$,
if there is a weak simulation $\mathcal{S}$ such that $p \; \mathcal{S} \; q$.

**Example:**
Prove that $Q = \tau.a.\tau.b.Q$ weakly simulates $P = a.b.P$.
Prove that $P = a.b.P$ weakly simulates $Q = \tau.a.\tau.b.Q$.



# Weak Bisimulation

**Definition:**(* straightforward / should be no surprise *)
A binary relation $\mathcal{B}$ is *a* **weak bisimulation**
if both $\mathcal{B}$ and its converse $\mathcal{B}^{-1}$ are weak simulations.

$P$ and $Q$ are **weakly bisimilar**, **weakly equivalent**, or **observation equivalent**, written $P \approx Q$,
if there exists a weak bisimulation $\mathcal{B}$ with $P \; \mathcal{B} \; Q$.

Alternatively:

$$\approx \stackrel{\text{def}}{=} \bigcup \{ \; \mathcal{B} \mid \mathcal{B} \text{ is weak bisimulation} \; \}$$

**Proposition:**

1. $\approx$ is itself a weak bisimulation.
2. $\approx$ is an equivalence relation.

1. every strong simulation is also a weak one
2. $P \sim Q$ implies $P \approx Q$

Examples ?

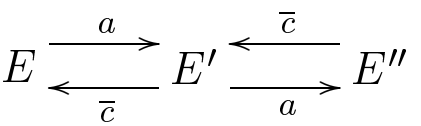
Proof ?

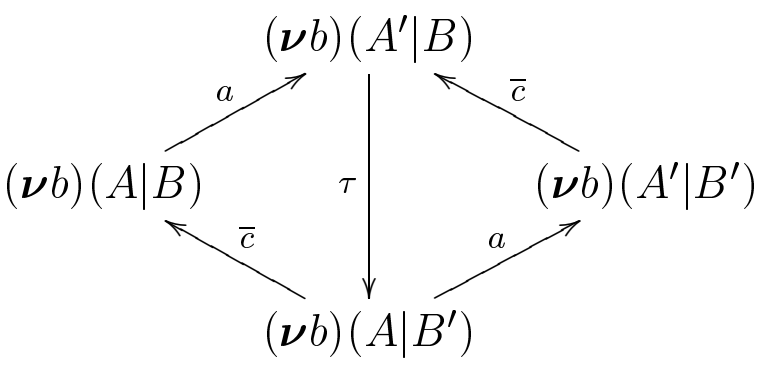


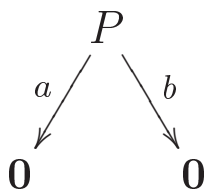
# Example

$$
\begin{array}{lcll}
A & \stackrel{\text{def}}{=} & a.A' & (= a.\overline{b}.A) \\
A' & \stackrel{\text{def}}{=} & \overline{b}.A & \\
B & \stackrel{\text{def}}{=} & b.B' & (= b.\overline{c}.B) \\
B' & \stackrel{\text{def}}{=} & \overline{c}.B &
\end{array}
\qquad
\begin{array}{lcl}
E & \stackrel{\text{def}}{=} & a.E' \\
E' & \stackrel{\text{def}}{=} & a.E'' + \overline{c}.E \\
E'' & \stackrel{\text{def}}{=} & \overline{c}.E'
\end{array}
$$

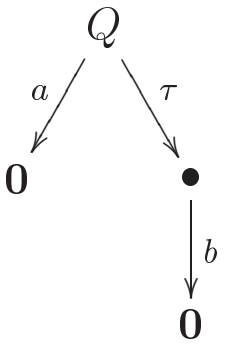
Prove that $(\boldsymbol{\nu} b)(A|B) \approx E$.



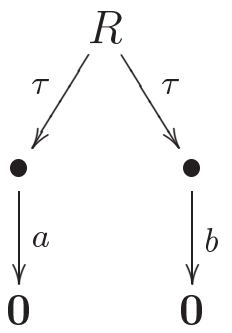






$P = a + b$




$Q = a + \tau.b$

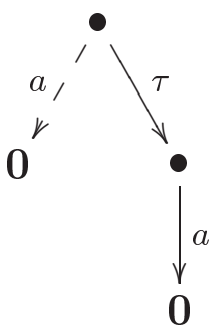



$R = \tau.a + \tau.b$

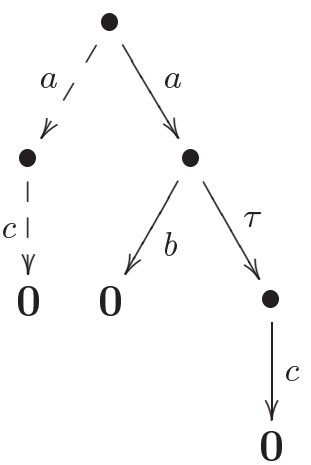


# Some Equivalences

$\tau.a \approx a$




$a + \tau.a \approx \tau.a$




$a.c + a.(b + \tau.c)$
$\approx a.(b + \tau.c)$

**Theorem:**
Let $P$ be any process.
Let $N, M$ any summations. Then:

1. $P \approx \tau.P$
2. $M + N + \tau.N \approx M + \tau.N$
3. $M + \alpha.P + \alpha(\tau.P + N) \approx M + \alpha(\tau.P + N)$



# Congruence Properties

**Proposition:**
Weak bisimilarity is a process congruence, i.e., . . .

**Example:**

- Observe $b \approx \tau.b$ !
- Let $C[\cdot] = a + [\cdot]$.

  Compare $C[b] = \boxed{a + b \stackrel{?}{\approx} a + \tau.b} = C[\tau.b]$ !

# Two Place Buffers

| | | |
|---|---|---|
| $\mathsf{Buff}_s^{(1)}(\,\vec{a}\,)$ | : | 1-place buffer containing $s$, where $\vec{a} = \{\mathsf{in}, \mathsf{out}\}$ |
| $\mathsf{Buff}_\epsilon^{(1)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $\mathsf{in}(x).\mathsf{Buff}_x^{(1)}\langle\,\vec{a}\,\rangle$ |
| $\mathsf{Buff}_v^{(1)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $\overline{\mathsf{out}}\langle v\rangle.\mathsf{Buff}_\epsilon^{(1)}\langle\,\vec{a}\,\rangle$ |
| $\mathsf{Buff}_s^{(2)}(\,\vec{a}\,)$ | : | 2-place buffer containing $s$ — SPECIFICATION |
| $\mathsf{Buff}_\epsilon^{(2)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $\mathsf{in}(x).\mathsf{Buff}_x^{(2)}\langle\,\vec{a}\,\rangle$ |
| $\mathsf{Buff}_v^{(2)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $\overline{\mathsf{out}}\langle v\rangle.\mathsf{Buff}_\epsilon^{(2)}\langle\,\vec{a}\,\rangle + \mathsf{in}(w).\mathsf{Buff}_{wv}^{(2)}\langle\,\vec{a}\,\rangle$ |
| $\mathsf{Buff}_{wv}^{(2)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $\overline{\mathsf{out}}\langle w\rangle.\mathsf{Buff}_v^{(2)}\langle\,\vec{a}\,\rangle$ |
| $\mathsf{Bluff}_s^{(2)}(\,\vec{a}\,)$ | : | 2-place buffer containing $s$ — IMPLEMENTATION |
| $\mathsf{Bluff}_\epsilon^{(2)}(\,\vec{a}\,)$ | $\stackrel{\text{def}}{=}$ | $(\boldsymbol{\nu}\mathsf{x})\left(\,\mathsf{Buff}^{(1)}\langle\,\mathsf{in},\mathsf{x}\,\rangle \mid \mathsf{Buff}^{(1)}\langle\,\mathsf{x},\mathsf{out}\,\rangle\,\right)$ |

☐ prove that $\mathsf{Buff}_\epsilon^{(2)}\langle\,\vec{a}\,\rangle \approx \mathsf{Bluff}_\epsilon^{(2)}\langle\,\vec{a}\,\rangle$



# Unique Solution of Equations

**Theorem:**

Let $\vec{X} = X_1, X_2, \ldots$ be a (possibly infinite) sequence of process variables. In the equations

$$\begin{array}{lcl} X_1 & \approx & \alpha_{11}.X_{k(11)} + \cdots + \alpha_{1n_1}.X_{k(1n_1)} \\ X_2 & \approx & \alpha_{21}.X_{k(11)} + \cdots + \alpha_{2n_1}.X_{k(2n_1)} \\ \cdots & \approx & \cdots \end{array}$$

assume that $\alpha_{ij} \neq \tau$. Then, up to $\approx$, there is a unique sequence $P_1, P_2, \ldots$ of processes which satisfies the equations.