# Data-flow analysis

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(based on material by Michael Schwartzbach and Erik Stenman)

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# Introduction to data-flow analysis

# Data-flow analysis

**Data-flow analysis** is a global analysis framework which can be used to compute – or, more precisely, approximate – various properties of programs.

The results of those analysis can be used to perform several optimisations like common sub-expression elimination, dead-code elimination, constant propagation, register allocation, etc.

# Example of data-flow analysis: liveness

A variable is said to be **live** at a given point if its value will be read later. While liveness is clearly undecidable, a conservative approximation can be computed using data-flow analysis.

This approximation can then be used, for example, to allocate registers: a set of variables which are never live at the same time can share a single register.

# Requirements of data-flow analysis

Data-flow analysis requires the program to be represented as a control flow graph (CFG).

To compute properties about the program, it assigns values to the nodes of the CFG. Those values must be related to each other by a special kind of partial order called a lattice.

We therefore start by introducing control flow graphs and lattice theory.

# Control flow graphs

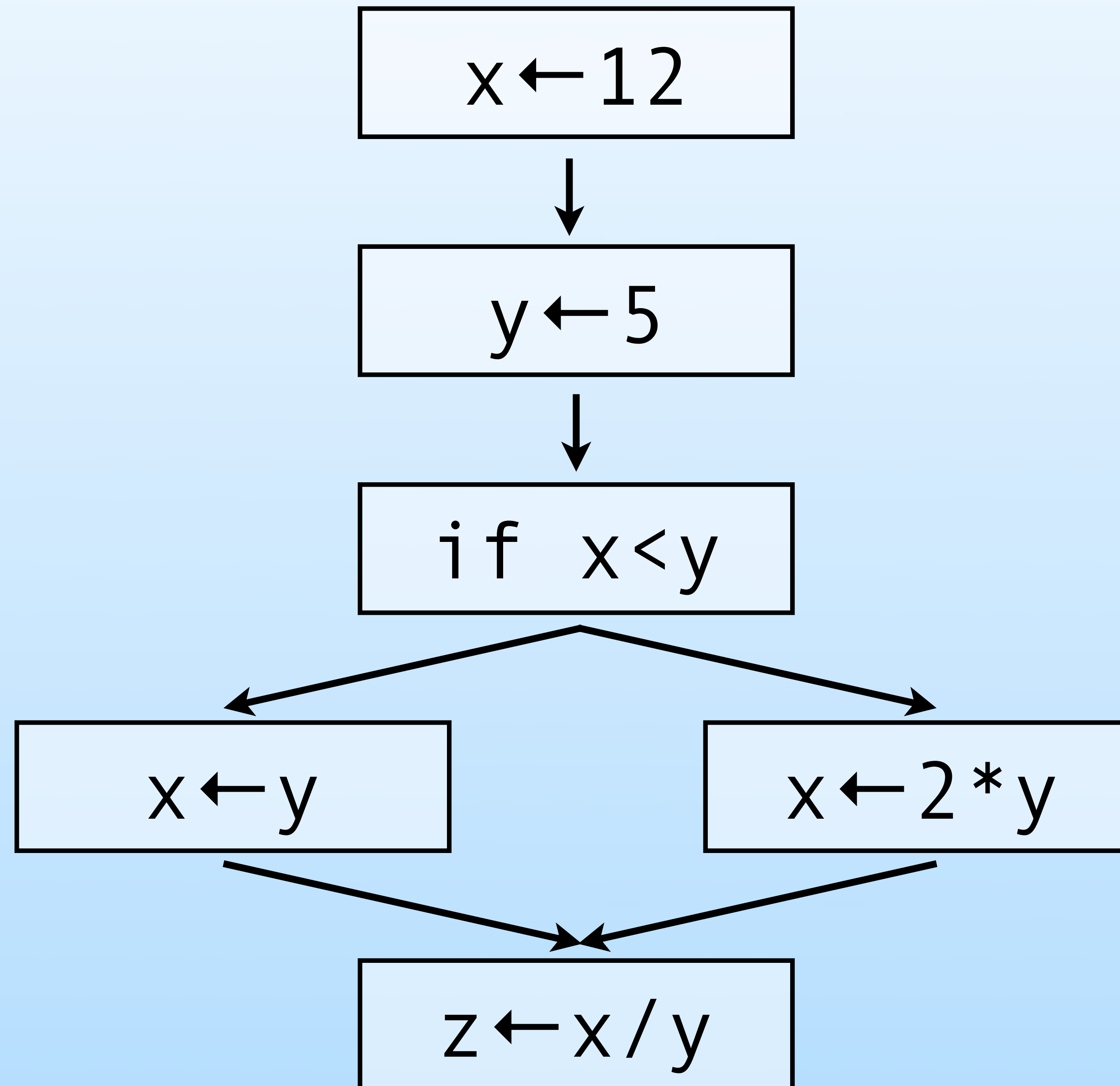
# Control flow graph

A **control flow graph** (**CFG**) is a graphical representation of a program.

The nodes of the CFG are the statements of that program.

The edges of the CFG represent the flow of control: there is an edge from $n_1$ to $n_2$ if and only if control can flow immediately from $n_1$ to $n_2$. That is, if the statements of $n_1$ and $n_2$ can be executed in direct succession.

# CFG example



# Predecessors and successors

In the CFG, the set of the immediate predecessors of a node *n* is written pred(*n*).

Similarly, the set of the immediate successors of a node *n* is written succ(*n*).

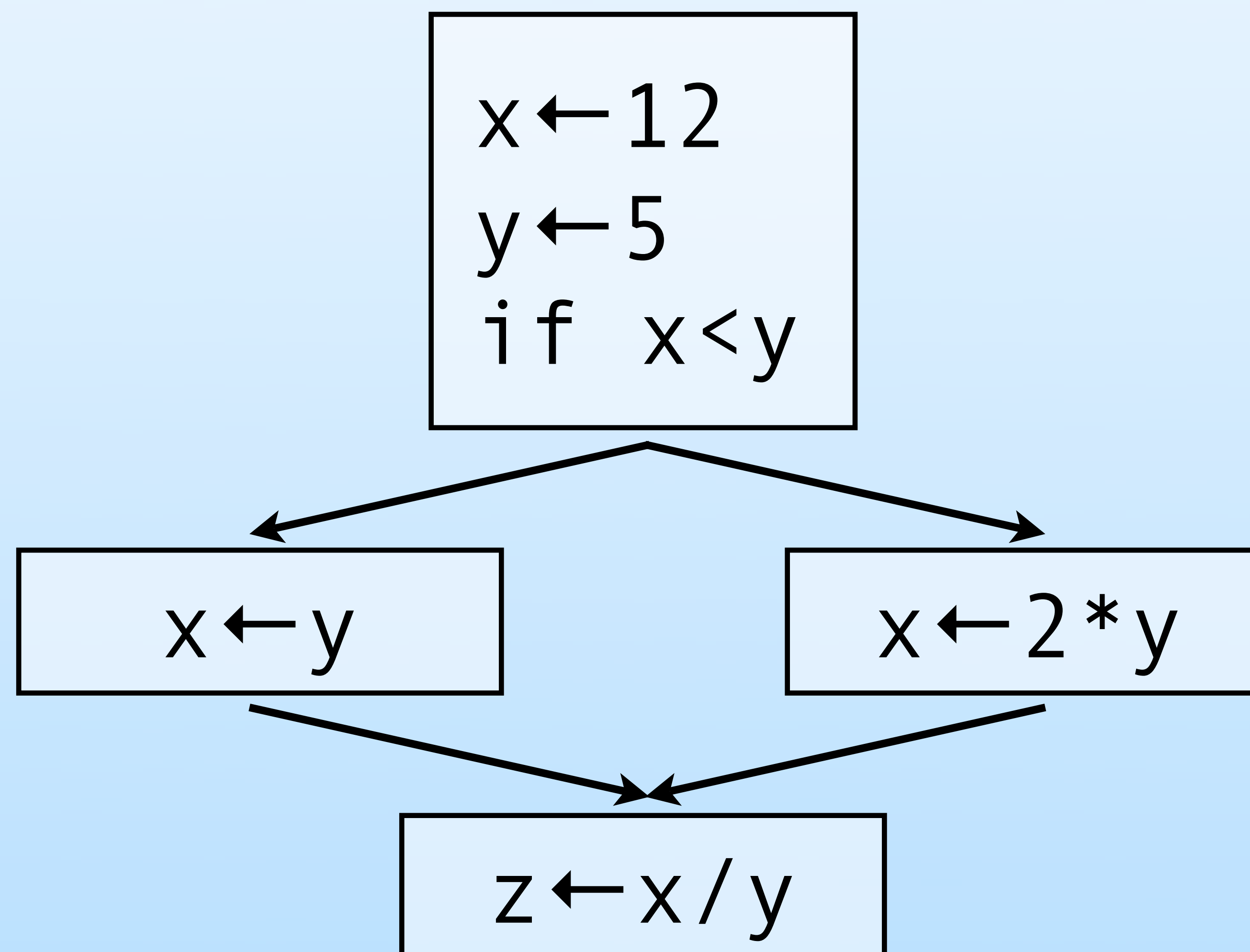
# Basic block

A **basic block** is a maximal sequence of statements for which control flow is purely linear.

That is, control always enters a basic block from the top – its first instruction – and leaves from the bottom – its last instruction.

Basic blocks are often used as the nodes of a CFG, in order to reduce its size.

# CFG example (nodes are basic blocks)



# Lattice theory

# Partial order

A **partial order** is a mathematical structure $(S, \sqsubseteq)$ composed of a set $S$ and a binary relation $\sqsubseteq$ on $S$, satisfying the following conditions:

- reflexivity: $\forall x \in S,\ x \sqsubseteq x$
- transitivity: $\forall x,y,z \in S,\ x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$
- anti-symmetry: $\forall x,y \in S,\ x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

# Partial order example

In Java, the set of types along with the sub-typing relation form a partial order.

According to that order, the type `String` is smaller (i.e. a sub-type) of the type `Object`.

The type `String` and `Integer` are not comparable: none of them is a sub-type of the other.

# Upper bound

Given a partial order $(S, \sqsubseteq)$ and a set $X \subseteq S$, $y \in S$ is an **upper bound** for $X$, written $X \sqsubseteq y$, if

$$\forall x \in X, x \sqsubseteq y.$$

A **least upper bound** (l.u.b.) for $X$, written $\sqcup X$, is defined by:

$$X \sqsubseteq \sqcup X \wedge \forall y \in S, X \sqsubseteq y \Rightarrow \sqcup X \sqsubseteq y$$

Notice that a least upper bound does *not* always exist.

# Lower bound

Given a partial order $(S, \sqsubseteq)$ and a set $X \subseteq S$, $y \in S$ is an **lower bound** for $X$, written $y \sqsubseteq X$, if

$$\forall x \in X, y \sqsubseteq x.$$

A **greatest lower bound** for $X$, written $\sqcap X$, is defined by:

$$\sqcap X \sqsubseteq X \wedge \forall y \in S, y \sqsubseteq X \Rightarrow y \sqsubseteq \sqcap X$$

Notice that a greatest lower bound does *not* always exist.

# Lattice

A **lattice** is a partial order $L = (S, \sqsubseteq)$ for which $\sqcup X$ and $\sqcap X$ exist for all $X \subseteq S$.

A lattice has a unique greatest element, written $\top$ and pronounced "**top**", defined as $\top = \sqcup S$.

It also has a unique smallest element, written $\bot$ and pronounced "**bottom**", defined as $\bot = \sqcap S$.

The **height** of a lattice is the length of the longest path from $\bot$ to $\top$.

# Finite partial orders and lattices

A partial order $(S,\sqsubseteq)$ is **finite** if the set $S$ contains a finite number of elements.

For such partial orders, the lattice requirements reduce to the following:

- $\top$ and $\bot$ exist,
- every pair of elements $x,y$ in S has a least upper bound – written $x \sqcup y$ – as well as a greatest lower bound – written $x \sqcap y$.

# Cover relation

In a partial order $(S,\sqsubseteq)$, we say that an element $y$ **covers** another element $x$ if:

$$(x \sqsubset y) \land (\forall z \in S, x \sqsubseteq z \sqsubset y \Rightarrow x = z)$$

where $x \sqsubset y \Leftrightarrow x \sqsubseteq y \land x \neq y$.

Intuitively, $y$ covers $x$ if $y$ is the smallest element greater than $x$.

# Hasse diagrams

A partial order can be represented graphically by a **Hasse diagram**.

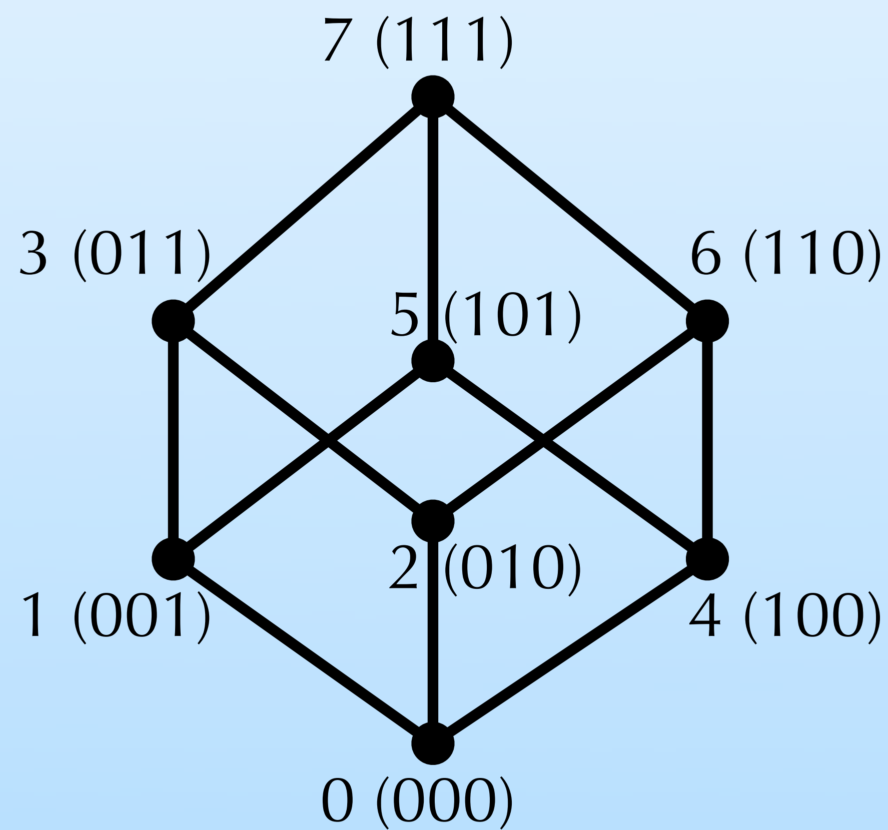
In such a diagram, the elements of the set are represented by dots.

If an element $y$ covers an element $x$, then the dot of $y$ is placed above the dot of $x$, and a line is drawn to connect the two dots.

# Hasse diagram example

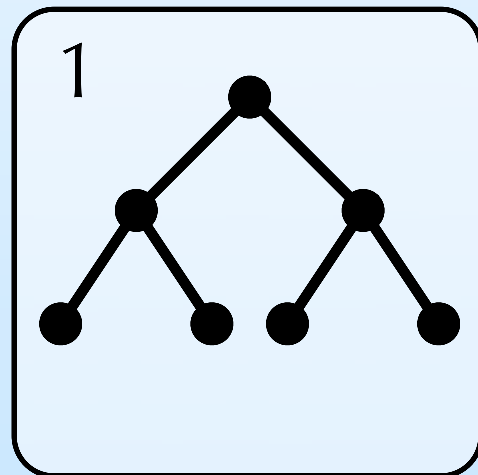
Hasse diagram for partial order $(S, \sqsubseteq)$ where
$S = \{ 0, 1, \ldots, 7 \}$ and $x \sqsubseteq y \Leftrightarrow (x \,\&\, y) = x$

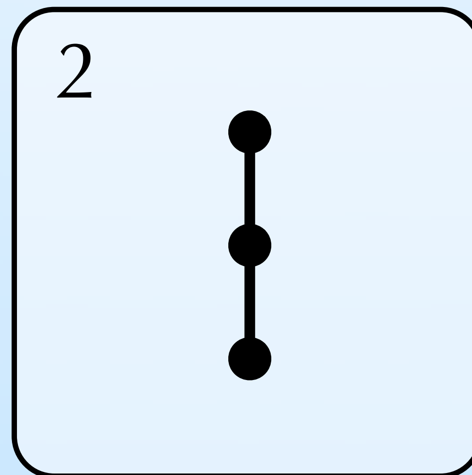
bit-wise and

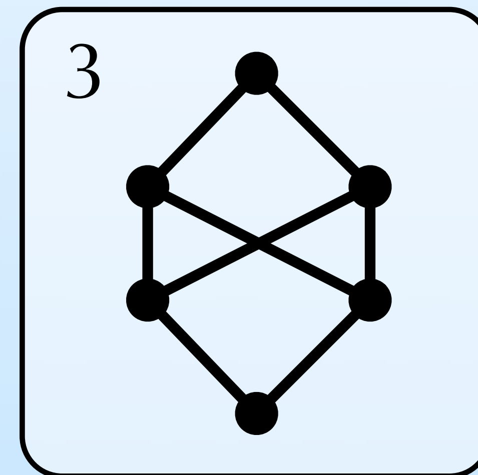


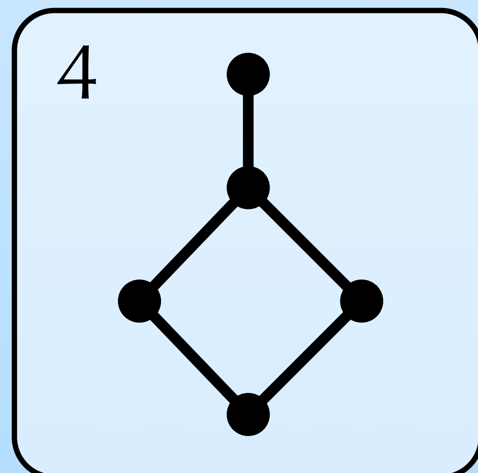
# Partial order examples

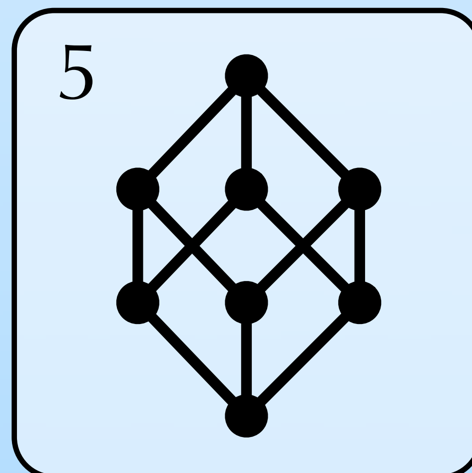
Which of the following partial orders are lattices?

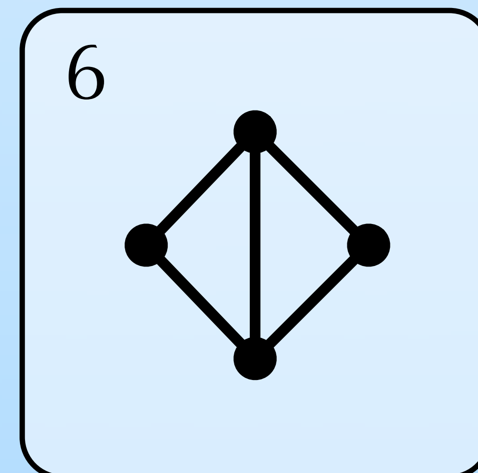


# Fixed points

# Monotone function

A function $f : L \to L$ is **monotone** if and only if:

$\forall x,y \in S,\ x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

This does *not* imply that $f$ is increasing, as constant functions are also monotone.

Viewed as functions, $\sqcap$ and $\sqcup$ are monotone in both arguments.

# Fixed point theorem

Definition: a value $v$ is a **fixed point** of a function $f$ if and only if $f(v) = v$.

Fixed point theorem: In a lattice $L$ with finite height, every monotone function $f$ has a unique least fixed point fix($f$), and it is given by:

$$\text{fix}(f) = \bot \sqcup f(\bot) \sqcup f^2(\bot) \sqcup f^3(\bot) \sqcup \ldots$$

# Fixed points and equations

Fixed points are interesting as they enable us to solve systems of equations of the following form:

$x_1 = F_1(x_1, \ldots, x_n)$
$x_2 = F_2(x_1, \ldots, x_n)$
…
$x_n = F_n(x_1, \ldots, x_n)$

where $x_i$ are variables, and $F_i : L^n \rightarrow L$ are monotone functions.

# Solving equation systems

An equation system like the one just presented has a unique least solution which is the least fixed point of the composite function $F : L^n \to L^n$ defined as:

$$F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$$

# Solving inequation systems

Systems of inequations of the following form:

$$x_1 \sqsubseteq F_1(x_1, \ldots, x_n)$$
$$x_2 \sqsubseteq F_2(x_1, \ldots, x_n)$$
$$\ldots$$
$$x_n \sqsubseteq F_n(x_1, \ldots, x_n)$$

can be solved similarly by observing that
$x \sqsubseteq y \Leftrightarrow x = x \sqcap y$ and rewriting the inequations.

# Data-flow analysis

# Data-flow analysis overview

Data-flow analysis works on a control-flow graph and a lattice $L$. The lattice can either be fixed for all programs, or depend on the analysed one.

A variable $v_n$ ranging over the values of $L$ is attached to every node $n$ of the CFG.

A set of (in)equations for these variables are then extracted from the CFG – according to the analysis being performed – and solved using the fixed point technique.

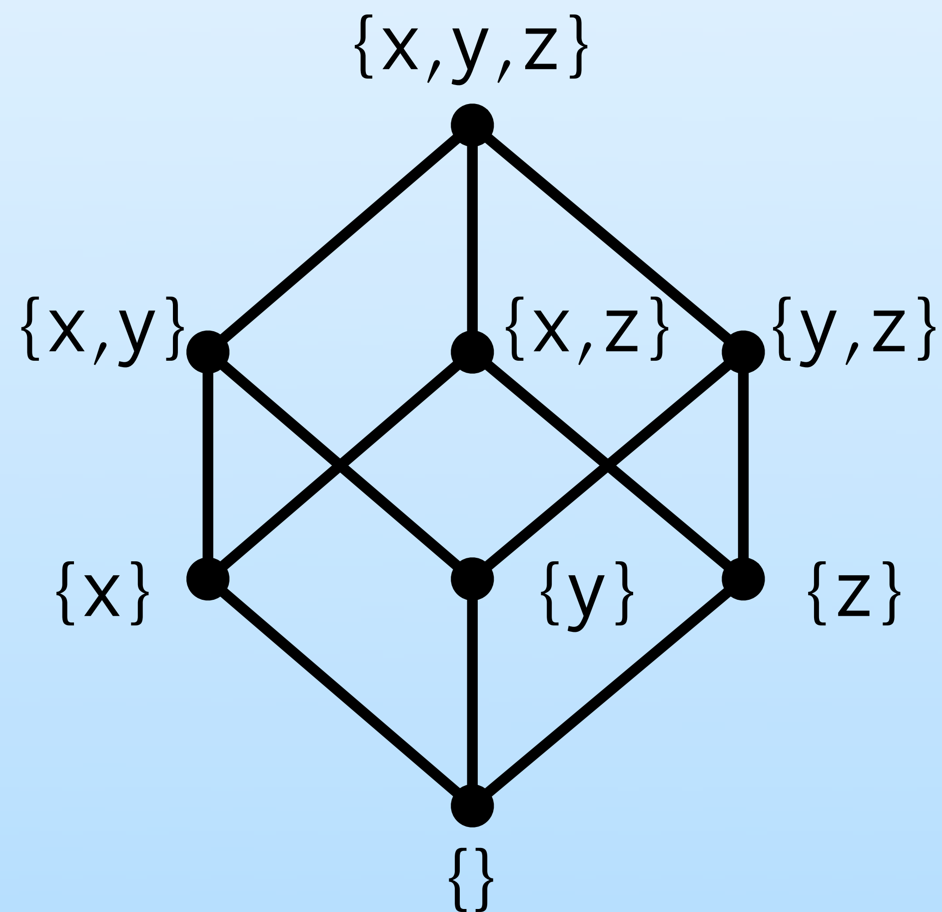
# Data-flow analysis example: liveness

As we have seen, liveness is a property which can be approximated using data-flow analysis.

The lattice to use in that case is $L = \{ \mathcal{P}(V), \subseteq \}$ where $V$ is the set of variables appearing in the analysed program, and $\mathcal{P}$ is the power set operator (set of all subsets).

# Data-flow analysis example: liveness

For a program containing three variables x, y and z, the lattice for liveness is the following:



# Data-flow analysis example: liveness

To every node $n$ in the CFG, we attach a variable $v_n$ giving the set of variables live *before* that node.

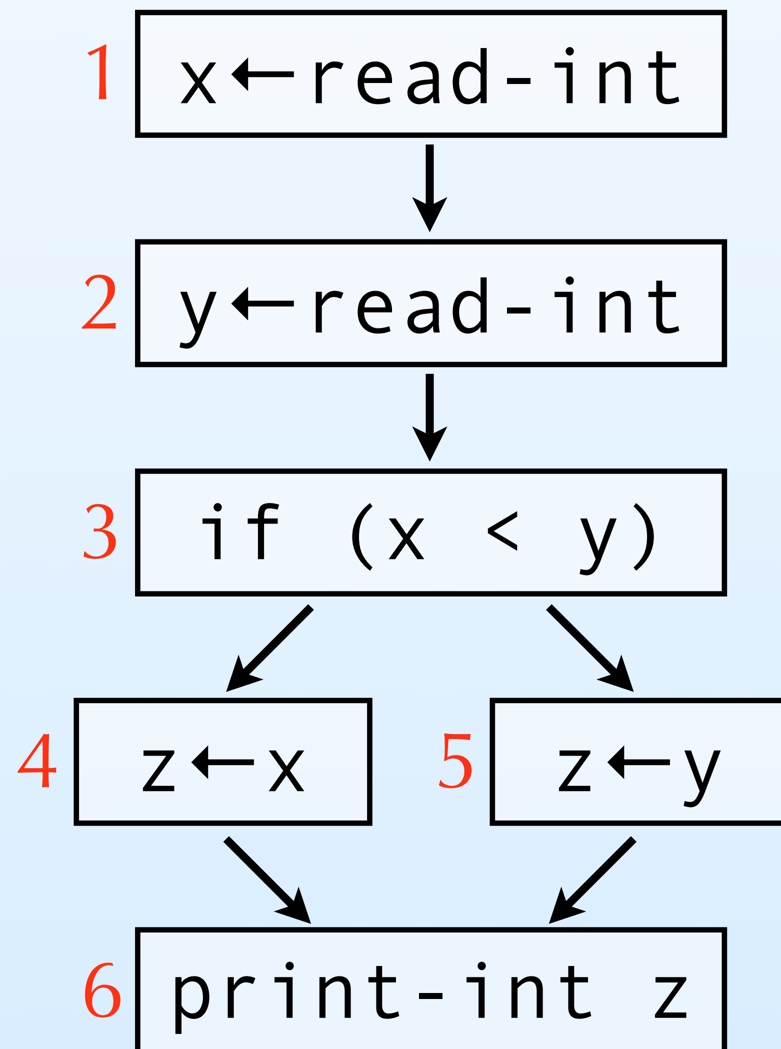
The value of that variable is given by:

$$v_n = (v_{s1} \cup v_{s2} \cup \ldots \setminus \text{written}(n)) \cup \text{read}(n)$$

where $s_1$, $s_2$, … are the successors of $n$, read($n$) is the set of program variables read by $n$, and written($n$) is the set of variables written by $n$.

# Data-flow analysis example: liveness

CFG




constraints

$v_1 = v_2 \setminus \{ x \}$

$v_2 = v_3 \setminus \{ y \}$

$v_3 = v_4 \cup v_5 \cup \{ x, y \}$

$v_4 = v_6 \cup \{ x \} \setminus \{ z \}$

$v_5 = v_6 \cup \{ y \} \setminus \{ z \}$

$v_6 = \{ z \}$

solution

$v_1 = \{ \}$

$v_2 = \{ x \}$

$v_3 = \{ x, y \}$

$v_4 = \{ x \}$

$v_5 = \{ y \}$

$v_6 = \{ z \}$

# Fixed point algorithm

To solve the data-flow constraints, we construct the composite function $F$ and compute its least fixed point by iteration.

$F(x_1, x_2, x_3, x_4, x_5, x_6) =$
$(x_2 \setminus \{x\}, x_3 \setminus \{y\}, x_4 \cup x_5 \cup \{x,y\}, x_6 \cup \{x\} \setminus \{z\}, x_6 \cup \{y\} \setminus \{z\}, \{z\})$

| Iteration | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 0 | { } | { } | { } | { } | { } | { } |
| 1 | { } | { } | { x, y } | { x } | { y } | { z } |
| 2 | { } | { x } | { x, y } | { x } | { y } | { z } |
| 3 | { } | { x } | { x, y } | { x } | { y } | { z } |

# Work-list algorithm

Computing the fixed point by simple iteration as we did works, but is wasteful as the information for *all* nodes is re-computed at every iteration.

It is possible to do better by remembering, for every variable $v$, the set dep($v$) of the variables whose value depends on the value of $v$ itself.

Then, whenever the value of some variable $v$ changes, we only re-compute the value of the variables which are in dep($v$).

# Work-list algorithm

$x_1 = x_2 = \ldots = x_n = \bot$
$q = [\ v_1, \ldots, v_n\ ]$
**while** $(q \neq [])$
  assume $q = [\ v_i, \ldots\ ]$
  $y = F_i\,(x_1, \ldots, x_n)$
  $q = q$.tail
  **if** $(y \neq x_i)$
    **for** $(v \in \text{dep}(v_i))$
      **if** $(v \notin q)$ $q$.append$(v)$
    $x_i = y$

# Work-list algorithm liveness example

| $q$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $[v_1,v_2,v_3,v_4,v_5,v_6]$ | {} | {} | {} | {} | {} | {} |
| $[v_2,v_3,v_4,v_5,v_6]$ | {} | {} | {} | {} | {} | {} |
| $[v_3,v_4,v_5,v_6]$ | {} | {} | {} | {} | {} | {} |
| $[v_4,v_5,v_6,v_2]$ | {} | {} | {x,y} | {} | {} | {} |
| $[v_5,v_6,v_2,v_3]$ | {} | {} | {x,y} | {x} | {} | {} |
| $[v_6,v_2,v_3,v_3]$ | {} | {} | {x,y} | {x} | {y} | {} |
| $[v_2,v_3,v_4,v_5]$ | {} | {} | {x,y} | {x} | {y} | {z} |
| $[v_3,v_4,v_5,v_1]$ | {} | {x} | {x,y} | {x} | {y} | {z} |
| $[v_4,v_5,v_1]$ | {} | {x} | {x,y} | {x} | {y} | {z} |
| $[v_5,v_1]$ | {} | {x} | {x,y} | {x} | {y} | {z} |
| $[v_1]$ | {} | {x} | {x,y} | {x} | {y} | {z} |
| [] | {} | {x} | {x,y} | {x} | {y} | {z} |

# Work-list algorithm improvements

In our liveness example, the work-list algorithm would have terminated in only six iterations if the initial queue had been reversed!

This is due to the fact that liveness analysis is a **backward** analysis: the value of variable $v_n$ depends on the successors of $n$. For such analysis, it is better to organise the queue with the latest nodes first.

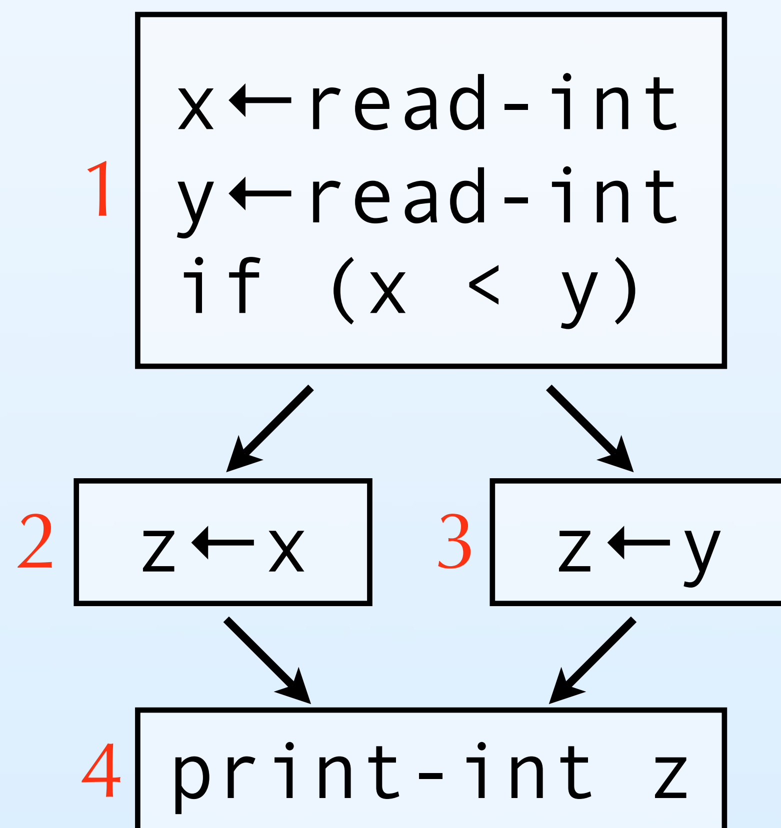
# Working with basic blocks

Until now, we considered that the CFG nodes were single instructions. In practice, basic blocks tend to be used as nodes, to reduce the size of the CFG.

When data-flow analysis is performed on a CFG composed of basic blocks, a variable is attached to every block, and not to every instruction. Computing the result of the analysis for individual instructions is however trivial.

# Working with basic blocks: liveness example

CFG




constraints

$v_1 = v_2 \cup v_3 \setminus \{ x, y \}$

$v_2 = v_4 \cup \{ x \} \setminus \{ z \}$

$v_3 = v_4 \cup \{ y \} \setminus \{ z \}$

$v_4 = \{ z \}$

solution

$v_1 = \{ \}$

$v_2 = \{ x \}$

$v_3 = \{ y \}$

$v_4 = \{ z \}$

# Summary

Data-flow analysis is a framework which can be used to compute approximations of various properties about programs. This is done by solving a set of equations with monotonous right-hand sides, using the fixed point algorithm.

As a first example, we have examined the computation of an approximation of liveness. More examples will follow.