Foundations of Dataflow Analysis

Advanced Compiler Techniques
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Dataflow Analysis

Compile-Time Reasoning About

- Run-Time Values of Variables or Expressions at different program points:
  - Which assignment statements produced the value of the variables at this point?
  - Which variables contain values that are no longer used after this program point?
  - What is the range of possible values of a variable at this program point?

Dataflow Analysis

- Assumptions:
  - We have a syntactically and semantically correct program (as far as compile time analysis can determine this).
  - We have the “whole” program, or a clearly defined subset of the program which will only interact with the rest of the program through a predefined interface.

    (That is, no *self* modifying code, and if the interface is a function then the parameters can take any value of the given type.)

Dataflow Analysis: Basic Idea

- Information about a program represented using values from an algebraic structure called *lattice*. (We will call this set of values $\mathbb{P}$.)
- Analysis produces a lattice value for each program point.
- Two flavors of analysis:
  - *Forward dataflow analyses.*
  - *Backward dataflow analyses.*

Forward Dataflow Analysis

- Analysis propagates values forward through control flow graph with flow of control
  - Each node has a transfer function *f*
    - Input – value at program point before node.
    - Output – new value at program point after node.
  - Values flow from program points after predecessor nodes to program points before successor nodes.
  - At join points, values are combined using a merge function.
- Canonical Example: Reaching Definitions.

Backward Dataflow Analysis

- Analysis propagates values backward through control flow graph against flow of control:
  - Each node has a transfer function *f*
    - Input – value at program point after node.
    - Output – new value at program point before node.
  - Values flow from program points before successor nodes to program points after predecessor nodes.
  - At split points, values are combined using a merge function.
- Canonical Example: Live Variables.

Partial Orders

- Set $\mathbb{P}$
- Partial order $\leq$ such that $\forall\, x, y, z \in \mathbb{P}$

i. $x \leq x$ (reflexive)

ii. $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetric)

iii. $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitive)

Upper Bounds

- If $\mathbb{S} \subseteq \mathbb{P}$ then
  - $x \in \mathbb{P}$ is an *upper bound* of $\mathbb{S}$ if $\forall y \in \mathbb{S}, y \leq x$
  - $x \in \mathbb{P}$ is the *least upper bound* (lub) of $\mathbb{S}$ if
    - x is an upper bound of $\mathbb{S}$, and
    - $x \leq y$ for all upper bounds y of $\mathbb{S}$
  - $\vee$ - *join*, least upper bound, supremum (sup)
    - $\bigvee \mathbb{S}$ is the least upper bound of $\mathbb{S}$
    - $x \vee y$ is the least upper bound of $\{x, y\}$

Lower Bounds

- If $\mathbb{S} \subseteq \mathbb{P}$ then
  - $x \in \mathbb{P}$ is a *lower bound* of $\mathbb{S}$ if $\forall y \in \mathbb{S}$, $x \leq y$
  - $x \in \mathbb{P}$ is the *greatest lower bound* (glb) of $\mathbb{S}$ if
    - x is a lower bound of $\mathbb{S}$, and
    - $y \leq x$ for all lower bounds y of $\mathbb{S}$
  - $\wedge$ - *meet*, greatest lower bound, infimum (inf)
    - $\wedge$ S is the greatest lower bound of S
    - $x \wedge y$ is the greatest lower bound of $\{x, y\}$

Coverings

- Notation: $x < y$ if $x \leq y$ and $x \neq y$
- x is *covered by* y (y *covers* x) if
  - $x < y$, and
  - $x \leq z < y \Rightarrow x = z$
- Conceptually, y covers x if there are no elements between x and y

Dataflow Analysis: Basic Idea

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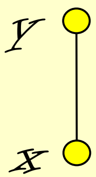
  - *Forward dataflow analyses.*
  - *Backward dataflow analyses.*

Hasse Diagram

- We can visualize a partial order with a Hasse Diagram.
- For each element x we draw a circle:

- If y covers x
  - Line from y to x
  - y above x in diagram

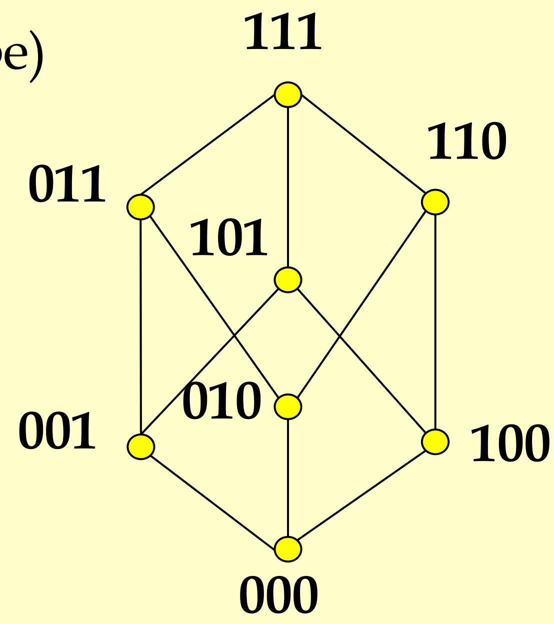


Hasse Diagram: Example

$\mathbb{P} = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$x \leq y$ if (x bitwise_and y) = x

(standard boolean lattice, also called hypercube)



Lattices

- If $x \wedge y$ and $x \vee y$ exist for all $x,y \in \mathbb{P}$, then $\mathbb{P}$ is a *lattice*.
- If $\bigwedge \mathbb{S}$ and $\bigvee \mathbb{S}$ exist for all $\mathbb{S} \subseteq \mathbb{P}$, then $\mathbb{P}$ is a *complete lattice*.
- Theorem: All finite lattices are complete.
- Example of a lattice that is not complete
  - Integers $\mathbb{Z}$
  - For any $x,y \in \mathbb{Z}$, $x \vee y = max(x,y)$, $x \wedge y = min(x,y)$
  - But $\bigvee \mathbb{Z}$ and $\bigwedge \mathbb{Z}$ do not exist
  - $\mathbb{Z} \cup \{+\infty, -\infty\}$ is a complete lattice

Top and Bottom

- Greatest element of $\mathbb{P}$ (if it exists) is *top* ($\top$).
- Least element of $\mathbb{P}$ (if it exists) is *bottom* ($\bot$).

Connection between $\leq$, $\wedge$, and $\vee$

The following 3 properties are equivalent:

- $x \leq y$
- $x \vee y = y$
- $x \wedge y = x$

- Will prove:
  - $x \leq y \Rightarrow x \vee y = y$ and $x \wedge y = x$
  - $x \vee y = y \Rightarrow x \leq y$
  - $x \wedge y = x \Rightarrow x \leq y$
- By Transitivity,
  - $x \vee y = y \Rightarrow x \wedge y = x$
  - $x \wedge y = x \Rightarrow x \vee y = y$

Connecting Lemma Proofs (1)

- Proof of $x \leq y \Rightarrow x \vee y = y$
  - $x \leq y \;\Rightarrow\; y$ is an upper bound of $\{x,y\}$.
  - Any upper bound z of $\{x,y\}$ must satisfy $y \leq z$.
  - So y is least upper bound of $\{x,y\}$ and $x \vee y = y$
- Proof of $x \leq y \Rightarrow x \wedge y = x$
  - $x \leq y \;\Rightarrow\; x$ is a lower bound of $\{x,y\}$.
  - Any lower bound z of $\{x,y\}$ must satisfy $z \leq x$.
  - So x is the greatest lower bound of $\{x,y\}$, that is $x \wedge y = x$

Connecting Lemma Proofs (2)

- Proof of $x \vee y = y \Rightarrow x \leq y$
  - y is an upper bound of $\{x,y\} \Rightarrow x \leq y$
- Proof of $x \wedge y = x \Rightarrow x \leq y$
  - x is a lower bound of $\{x,y\} \Rightarrow x \leq y$

Lattices as Algebraic Structures

- Have defined $\vee$ and $\wedge$ in terms of $\leq$.
- Now define $\leq$ in terms of $\vee$ and $\wedge$:
  - Start with $\vee$ and $\wedge$ as arbitrary algebraic operations that satisfy associative, commutative, idempotence, and absorption laws.
  - Will define $\leq$ using $\vee$ and $\wedge$.
  - Will show that $\leq$ is a partial order.

Algebraic Properties of Lattices

Assume arbitrary operations $\vee$ and $\wedge$ such that

- $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity of $\vee$)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (associativity of $\wedge$)
- $x \vee y = y \vee x$ (commutativity of $\vee$)
- $x \wedge y = y \wedge x$ (commutativity of $\wedge$)
- $x \vee x = x$ (idempotence of $\vee$)
- $x \wedge x = x$ (idempotence of $\wedge$)
- $x \vee (x \wedge y) = x$ (absorption of $\vee$ over $\wedge$)
- $x \wedge (x \vee y) = x$ (absorption of $\wedge$ over $\vee$)

Connection Between $\wedge$ and $\vee$

Theorem: $x \vee y = y$ if and only if $x \wedge y = x$

- Proof of $x \vee y = y \Rightarrow x = x \wedge y$

$x = x \wedge (x \vee y)$	(by absorption)
$= x \wedge y$	(by assumption)

- Proof of $x \wedge y = x \Rightarrow y = x \vee y$

$y = y \vee (y \wedge x)$	(by absorption)
$= y \vee (x \wedge y)$	(by commutativity)
$= y \vee x$	(by assumption)
$= x \vee y$	(by commutativity)

Properties of $\leq$

- Define $x \leq y$ if $x \vee y = y$
- Proof of transitive property. Show that

$$x \vee y = y \text{ and } y \vee z = z \Rightarrow x \vee z = z$$

$$\begin{aligned} x \vee z &= x \vee (y \vee z) && \text{(by assumption)} \\ &= (x \vee y) \vee z && \text{(by associativity)} \\ &= y \vee z && \text{(by assumption)} \\ &= z && \text{(by assumption)} \end{aligned}$$

Properties of $\leq$

- Proof of asymmetry property. Show that $x \vee y = y$ and $y \vee x = x \Rightarrow x = y$

$$
\begin{aligned}
x &= y \vee x && \text{(by assumption)} \\
&= x \vee y && \text{(by commutativity)} \\
&= y && \text{(by assumption)}
\end{aligned}
$$

- Proof of reflexivity property. Show that $x \vee x = x$

$$x \vee x = x \qquad \text{(by idempotence)}$$

Properties of $\leq$

- Induced operation $\leq$ agrees with original definitions of $\vee$ and $\wedge$, i.e.,
  - $x \vee y = \sup \{x, y\}$
  - $x \wedge y = \inf \{x, y\}$

Proof of $x \vee y = \sup \{x, y\}$

- Consider any upper bound u for x and y.
- Given $x \vee u = u$ and $y \vee u = u$,
  show $x \vee y \leq u$,
  i.e., $(x \vee y) \vee u = u$

$u = x \vee u$ (by assumption)

$= x \vee (y \vee u)$ (by assumption)

$= (x \vee y) \vee u$ (by associativity)

Proof of $x \wedge y = \inf \{x, y\}$

- Consider any lower bound l for x and y.
- Given $x \wedge l = l$ and $y \wedge l = l$,

  show $l \leq x \wedge y$,

  i.e., $(x \wedge y) \wedge l = l$

  $l = x \wedge l$ (by assumption)

  $= x \wedge (y \wedge l)$ (by assumption)

  $= (x \wedge y) \wedge l$ (by associativity)

Chains

- A set $\mathbb{S}$ is a *chain* if $\forall x,y \in \mathbb{S}.\ y \leq x$ or $x \leq y$
- $\mathbb{P}$ has no infinite chains if every chain in $\mathbb{P}$ is finite
- $\mathbb{P}$ satisfies the *ascending chain condition* if

for all sequences $x_1 \leq x_2 \leq \ldots$ there exists n such that $x_n = x_{n+1} = \ldots$

That is, all increasing sequences in $\mathbb{P}$ eventually becomes constant.

Dataflow Analysis (repetition)

- Information about a program represented using values from a *lattice* ($\mathbb{P}$). Analysis propagates values through control flow graph, either forwards or backwards.
- For forward analysis:
  - Each node has a transfer function f,
    - Input – value at program point before node.
    - Output – new value at program point after node.
  - Values flow from program points after predecessor nodes to program points before successor nodes.
  - At join points, values are combined using a merge function.

Transfer Functions

- Assume a lattice $\mathbb{P}$ of abstract values.
- Transfer function f: $\mathbb{P} \rightarrow \mathbb{P}$ for each node in control flow graph.
- f models the effect of the node on the program information.

Properties of Transfer Functions

Each dataflow analysis problem has a set $\mathbb{F}$ of transfer functions $f:\mathbb{P}\rightarrow\mathbb{P}$

- Identity function $i \in \mathbb{F}$
- $\mathbb{F}$ must be closed under composition: $\forall f, g \in \mathbb{F}$, the function $h = \lambda x. f(g(x)) \in \mathbb{F}$
- Each $f \in \mathbb{F}$ must be monotone: $x \leq y \Rightarrow f(x) \leq f(y)$
- Sometimes all $f \in \mathbb{F}$ are distributive: $f(x \vee y) = f(x) \vee f(y)$
- Distributivity $\Rightarrow$ monotonicity

Distributivity Implies Monotonicity

Proof:

- Assume $f(x \vee y) = f(x) \vee f(y)$
- Show: $x \vee y = y \Rightarrow f(x) \vee f(y) = f(y)$

$f(y) = f(x \vee y)$ (by assumption)

$= f(x) \vee f(y)$ (by distributivity)

Forward Dataflow Analysis

- Simulates forward execution of a program
- For each node n, we have

in_n – value at program point before n

out_n – value at program point after n

f_n – transfer function for n (given in_n, computes out_n)

- Require that solutions satisfy

i. $\forall n,\ out_n = f_n(in_n)$

ii. $\forall n \neq n_0,\ in_n = \vee \{\ out_m \mid m \in pred(n)\ \}$

iii. $in_{n0} = \perp$

Dataflow Equations

- Result is a set of dataflow equations

$$out_n := f_n(in_n)$$

$$in_n := \vee \{ out_m \mid m \in pred(n) \}$$

- Conceptually separates analysis problem from program.

Worklist Algorithm for Solving Forward Dataflow Equations

for each $n \in \mathbb{N}$ do $out_n := f_n(\bot)$

$worklist := \mathbb{N}$

while $worklist \neq \emptyset$ do:

  remove a node n from $worklist$

  $in_n := \vee \{ out_m \mid m \in pred(n) \}$

  $out_n := f_n(in_n)$

  if out_n changed then

    $worklist := worklist \cup succ(n)$

Correctness Argument

Why result satisfies dataflow equations?

- Whenever we process a node n,
  set $out_n := f_n(in_n)$
  Algorithm ensures that $out_n = f_n(in_n)$
- Whenever out_m changes, put *succ*(m) on worklist.
  Consider any node $n \in succ(m)$.
  It will eventually come off the worklist and the algorithm will set

$$in_n := \vee \{ out_m \mid m \in pred(n) \}$$

to ensure that $in_n = \vee \{ out_m \mid m \in pred(n) \}$

Termination Argument

Why does the algorithm terminate?

- Sequence of values taken on by in_n or out_n is a chain. If values stop increasing, the worklist empties and the algorithm terminates.
- If the lattice has the ascending chain property, the algorithm terminates
  - Algorithm terminates for finite lattices.
  - For lattices without the ascending chain property, we must use a *widening* operator.

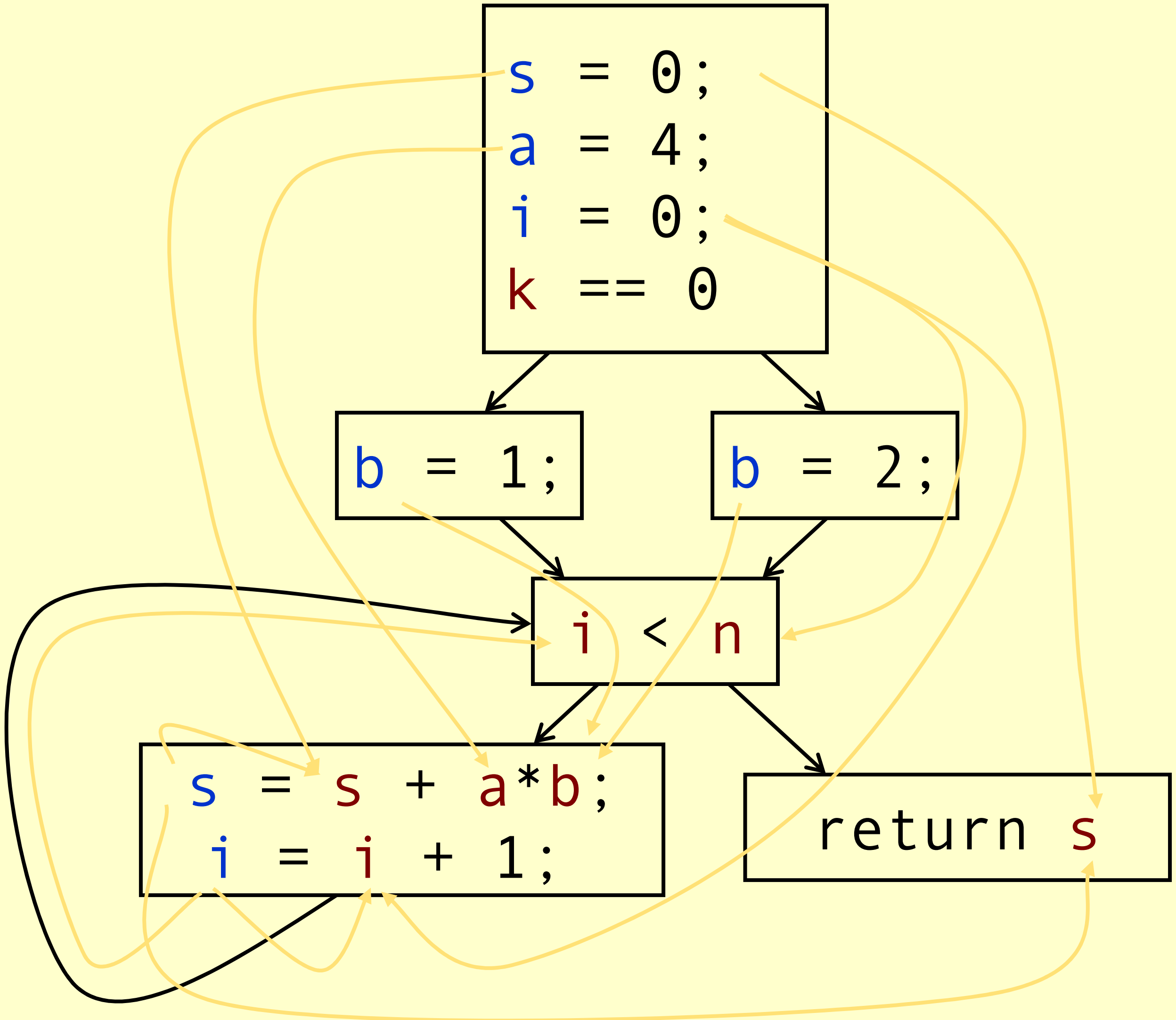
Widening Operators

- Detect lattice values that may be part of an infinitely ascending chain.
- Artificially raise value to least upper bound of the chain.
- Example:
  - Lattice is set of all subsets of integers.
  - Widening operator might raise all sets of size n or greater to TOP (the set of all integers).
  - Could be used to collect possible values taken on by a variable during execution of the program.

Reaching Definitions

- Concept of *definition* and *use*
  - `z = x+y`
    - is a definition of z
    - is a use of x and y
- A definition (d) reaches a use (u) if the value written by d may be read by u.

Reaching Definitions



Reaching Definitions Framework

- $\mathbb{P} = \wp$ (the powerset) of the set of definitions in the program (all subsets of the set of definitions).
- $\vee = \cup$ (order is $\subseteq$)
- $\perp = \varnothing$
- $\mathbb{F}$ = all functions f of the form $f(x) = a \cup (x - b)$
  - b is the set of definitions that the node kills.
  - a is the set of definitions that the node generates.

General pattern for many transfer functions

  - $f(x) = GEN \cup (x - KILL)$

Does Reaching Definitions Framework Satisfy Properties?

- $\subseteq$ satisfies conditions for $\leq$

  $x \subseteq y$ and $y \subseteq z \Rightarrow x \subseteq z$ (transitivity)

  $x \subseteq y$ and $y \subseteq x \Rightarrow y = x$ (asymmetry)

  $x \subseteq x$ (reflexivity)

- $\mathbb{F}$ satisfies transfer function conditions

  $\lambda x.\emptyset \cup (x - \emptyset) = \lambda x.x \in \mathbb{F}$ (identity)

  Will show $f(x \cup y) = f(x) \cup f(y)$ (distributivity)

$$
\begin{aligned}
f(x) \cup f(y) &= (a \cup (x - b)) \cup (a \cup (y - b)) \\
&= a \cup (x - b) \cup (y - b) \\
&= a \cup ((x \cup y) - b) \\
&= f(x \cup y)
\end{aligned}
$$

Does Reaching Definitions Framework Satisfy Properties?

What about composition?

- Given $f_1(x) = a_1 \cup (x-b_1)$ and $f_2(x) = a_2 \cup (x-b_2)$
- Show $f_1(f_2(x))$ can be expressed as $a \cup (x - b)$

$$
\begin{aligned}
f_1(f_2(x)) &= a_1 \cup ((a_2 \cup (x-b_2)) - b_1) \\
&= a_1 \cup ((a_2 - b_1) \cup ((x-b_2) - b_1)) \\
&= (a_1 \cup (a_2 - b_1)) \cup ((x-b_2) - b_1)) \\
&= (a_1 \cup (a_2 - b_1)) \cup (x-(b_2 \cup b_1))
\end{aligned}
$$

Let $a = (a_1 \cup (a_2 - b_1))$ and $b = b_2 \cup b_1$

Then $f_1(f_2(x)) = a \cup (x - b)$

General Result

All GEN/KILL transfer function frameworks satisfy the properties:

- Identity
- Distributivity
- Compositionality

Available Expressions Framework

- $\mathbb{P} = \wp$ (the powerset) of the set of all expressions in the program (all subsets of set of expressions).
- $\vee = \cap$ (order is $\supseteq$)
- $\perp = \wp$ (but $in_{n0} = \emptyset$)
- $\mathbb{F}$ = all functions f of the form
  $f(x) = a \cup (x-b)$.
  - b is set of expressions that node kills.
  - a is set of expressions that node generates.
- Another GEN/KILL analysis

Concept of Conservatism

- Reaching definitions use $\cup$ as join
  - Optimizations must take into account all definitions that reach along ANY path
- Available expressions use $\cap$ as join
  - Optimization requires expression to reach along ALL paths
- Optimizations must conservatively take all possible executions into account.
- Structure of analysis varies according to the way the results of the analysis are to be used.

Backward Dataflow Analysis

- Simulates execution of program backward against the flow of control.
- For each node n, we have

in_n – value at program point before n.

out_n – value at program point after n.

f_n – transfer function for n (given out_n, computes in_n).

- Require that solutions satisfy:

i. $\forall n.\ in_n = f_n(out_n)$

ii. $\forall n \notin \mathbb{N}_{final}.\ out_n = \vee \{\ in_m \mid m \in succ(n)\ \}$

iii. $\forall n \in \mathbb{N}_{final}.\ out_n = \bot$

Worklist Algorithm for Solving Backward Dataflow Equations

for each $n \in \mathbb{N}$ do $in_n := f_n(\bot)$

worklist := $\mathbb{N}$

while worklist $\neq \emptyset$ do

  remove a node n from worklist

  $out_n := \vee \{ in_m \mid m \in succ(n) \}$

  $in_n := f_n(out_n)$

  if in_n changed then

    worklist := worklist $\cup$ $pred(n)$

Live Variables Analysis Framework

- $\mathbb{P}$ = powerset of the set of all variables in the program (all subsets of the set of variables).
- $\vee = \cup$ (order is $\subseteq$)
- $\bot = \varnothing$
- $\mathbb{F}$ = all functions f of the form $f(x) = a \cup (x - b)$
  - b is set of variables that the node kills.
  - a is set of variables that the node reads.

Meaning of Dataflow Results

- Connection between executions of program and dataflow analysis results.
- Each execution generates a trajectory of states:
  - $s_0;s_1;\ldots;s_k$,where each $s_i \in \mathbb{S}$
- Map current state s_k to
  - Program point n where execution located.
  - Value x in dataflow lattice.
- Require $x \leq in_n$

Abstraction Function for Forward Dataflow Analysis

- Meaning of analysis results is given by an abstraction function $AF:\mathbb{S}\to\mathbb{P}$
- Require that for all states s

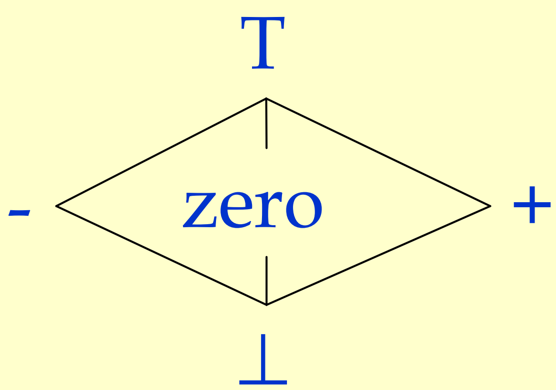
$$AF(s) \leq in_n$$

where n is the program point where the execution is located at in state s, and in_n is the abstract value before that point.

Sign Analysis Example

Sign analysis - compute sign of each variable v

- Base Lattice: flat lattice on {-,zero,+}



- Actual lattice records a value for each variable
  - Example element: [a→+, b→zero, c→-]

Interpretation of Lattice Values

If value of v in lattice is:

- ⊥: no information about the sign of v.
- -: variable v is negative.
- zero: variable v is 0 .
- +: variable v is positive.
- T: v may be positive or negative or 0.

Operation ⊗ on Lattice

⊗	⊥	-	zero	+	T
⊥	⊥	-	zero	+	T
-	-	+	zero	-	T
zero	zero	zero	zero	zero	zero
+	+	-	zero	+	T
T	T	T	zero	T	T

Transfer Functions

Defined by structural induction on the shape of nodes:

- If n of the form v = c
  - $f_n(x) = x[v \rightarrow +]$ if c is positive
  - $f_n(x) = x[v \rightarrow zero]$ if c is 0
  - $f_n(x) = x[v \rightarrow -]$ if c is negative
- If n of the form $v_1 = v_2 * v_3$
  - $f_n(x) = x[v_1 \rightarrow x[v_2] \otimes x[v_3]]$

Abstraction Function

- *AF*(s)[v] = sign of v
  - *AF*([a→5, b→0, c→-2]) = [a→**+**, b→**zero**, c→**-**]
- Establishes meaning of the analysis results
  - If analysis says a variable v has a given sign
  - then v always has that sign in actual execution.
- Two sources of imprecision
  - Abstraction Imprecision – concrete values (integers) abstracted as lattice values (**-**,**zero**, and **+**);
  - Control Flow Imprecision – one lattice value for all different flow of control possibilities.

Imprecision Example

Abstraction Imprecision:

[a→1] abstracted as [a→**+**]

[a→⊥, b→⊥, c→⊥]

```
a = 1
```

[a→**+**, b→⊥, c→⊥]   [a→**+**, b→⊥, c→⊥]

```
b = -1        b = 1
```

[a→**+**, b→**-**, c→⊥]   [a→**+**, b→**+**, c→⊥]

[a→**+**, b→**T**, c→⊥]

```
c = a*b
```

[a→**+**, b→**T**, c→**T**]

Control Flow Imprecision:

[b→T] summarizes results of all executions.

In any execution state s, *AF*(s)[b]≠T

General Sources of Imprecision

- Abstraction Imprecision
  - Lattice values less precise than execution values.
  - Abstraction function throws away information.
- Control Flow Imprecision
  - Analysis result has a single lattice value to summarize results of multiple concrete executions.
  - Join operation $\vee$ moves up in lattice to combine values from different execution paths.
  - Typically if $x \leq y$, then x is more precise than y.

Why Have Imprecision?

ANSWER: To make analysis tractable

- Conceptually infinite sets of values in execution.
  - Typically abstracted by finite set of lattice values.
- Execution may visit infinite set of states.
  - Abstracted by computing joins of different paths.

Augmented Execution States

- Abstraction functions for some analyses require augmented execution states.
  - Reaching definitions: states are augmented with the definition that created each value.
  - Available expressions: states are augmented with expression for each value.

Meet Over All Paths Solution

- What solution would be ideal for a forward dataflow analysis problem?
- Consider a path $p = n_0, n_1, \ldots, n_k, n$ to a node n
  (note that for all i, $n_i \in \mathit{pred}(n_{i+1})$)
- The solution must take this path into account:

$$f_p(\bot) = (f_{n_k}(f_{n_{k-1}}(\ldots f_{n_1}(f_{n_0}(\bot)) \ldots)) \leq in_n$$

- So the solution should have the property that

$$\wedge\{f_p(\bot) \mid p \text{ is a path to } n\} = in_n$$

Conservative Solution

- There is no algorithm to compute the optimal solution, due to infinite number of paths.
- A solution is conservative if for all paths p to n, $f_p(\perp) \leq in_n$

Soundness Proof of Analysis Algorithm

Property to prove:

For all paths p to n, $f_p(\bot) \leq in_n$

- Proof is by induction on the length of p.
  - Uses monotonicity of transfer functions.
  - Uses following lemma.

Lemma:

The worklist algorithm produces a solution such that

if $n \in pred(m)$ then $out_n \leq in_m$

(That is, what you get out of a predecessor is more precise than what will go in to the node, because precision may be lost by the join function.)

Proof

- Base case: p is of length 0
  - Then $p = n_0$ and $f_p(\bot) = \bot = in_{n_0}$
- Induction step:
  - Assume theorem for all paths of length k.
  - Show for an arbitrary path p of length k+1.

Induction Step Proof

- Given a path $p = n_0, \ldots, n_k, n$ show $(f_{n_k}(f_{n_{k-1}}(\ldots f_{n_1}(f_{n_0}(\bot)) \ldots)) \leq in_n$

  By induction assumption: (theorem holds for all paths of length k)

  $$(f_{n_{k-1}}(\ldots f_{n1}(f_{n0}(\bot)) \ldots)) \leq in_{n_k}$$

  Apply f_{n_k} to both sides:

  $$f_{n_k}(f_{n_{k-1}}(\ldots f_{n_1}(f_{n_0}(\bot)) \ldots) \quad ? \quad f_{n_k}(in_{n_k})$$

  By monotonicity: $(x \leq y \Rightarrow f(x) \leq f(y))$

  $$(f_{n_k}(f_{n_{k-1}}(\ldots f_{n_1}(f_{n_0}(\bot)) \ldots)) \leq f_{n_k}(in_{n_k})$$

  By definition of f_{n_k}: $(f_{n_k}(in_{n_k}) = out_{n_k})$

  $$(f_{n_k}(f_{n_{k-1}}(\ldots f_{n_1}(f_{n_0}(\bot)) \ldots)) \leq out_{n_k}$$

Distributivity

- Distributivity preserves precision.
- If framework is distributive, then the worklist algorithm produces a precis result:

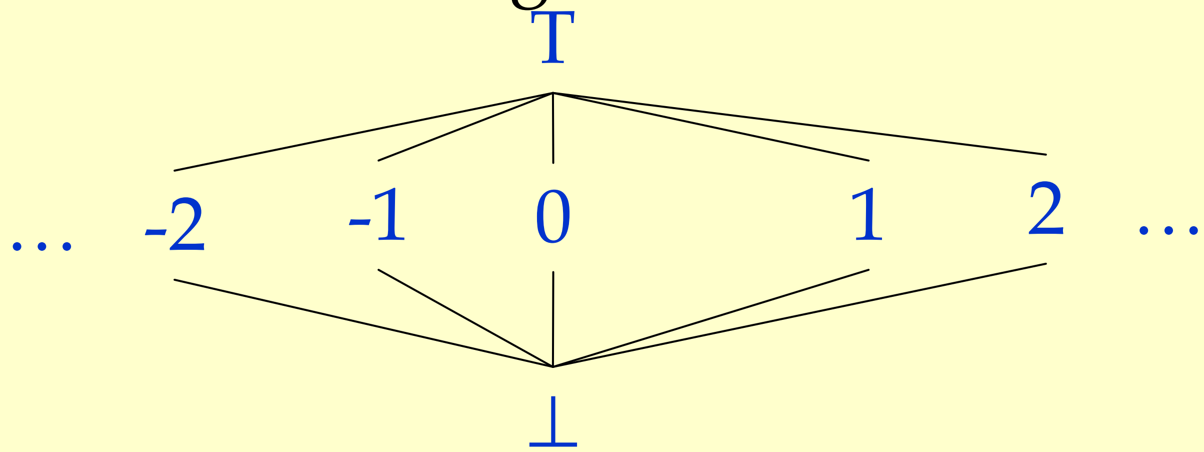
For all n:

$$\vee\{f_p(\bot) \mid p \text{ is a path to } n\} = in_n$$

Lack of Distributivity Example

Integer Constant Propagation (ICP)

- Flat lattice on integers

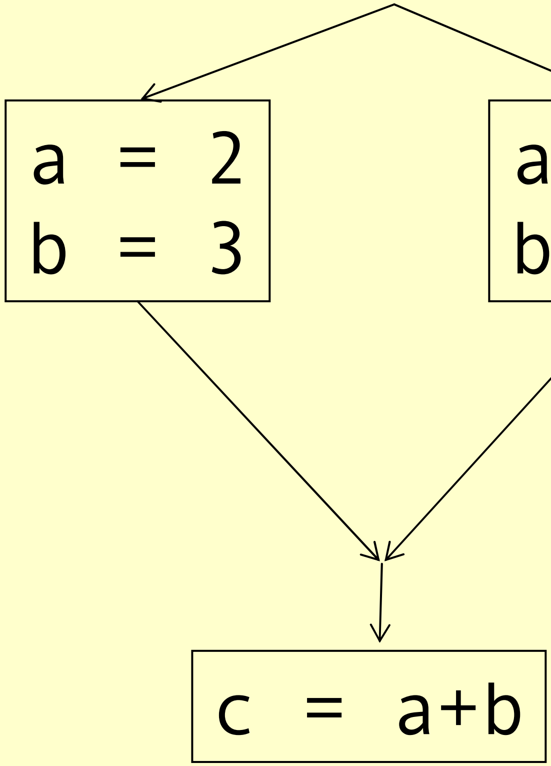


- Actual lattice records a value for each variable
  - Example element: [a→3, b→2, c→5]

Transfer Functions

- If n of the form v = c
  - $f_n(x) = x[v \to c]$
- If n of the form $v_1 = v_2 + v_3$
  - $f_n(x) = x[v_1 \to x[v_2] + x[v_3]]$

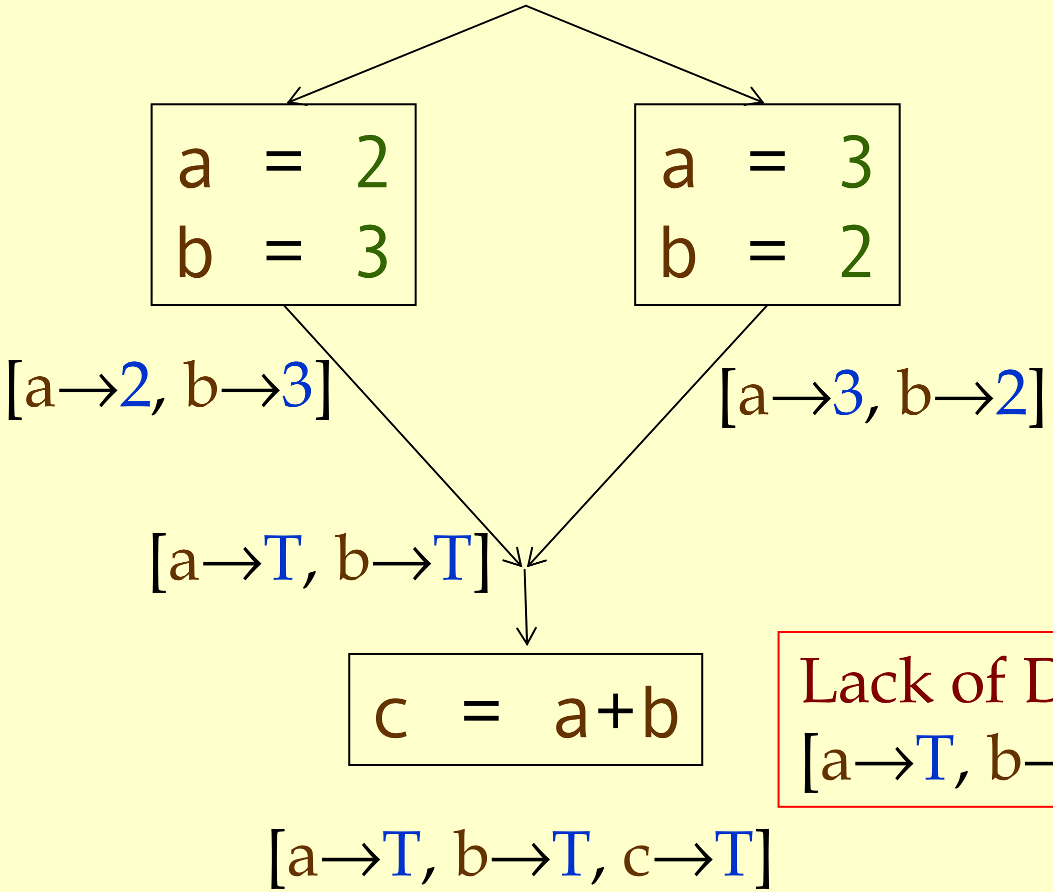
Lack of Distributivity Anomaly



Lack of distributivity of ICP

- Consider transfer function f for c = a + b
  (f(x) = x[c→x[a] + x[b]])
- f([a→3, b→2]) ∨ f([a→2, b→3]) =
  [a→3, b→2] [c→ [a→3, b→2][a] + [a→3, b→2][b]] ∨
  [a→2, b→3] [c→ [a→2, b→3][a] + [a→2, b→3][b]] =
  [a→3, b→2] [c→ 3 + 2] ∨ [a→2, b→3] [c→ 2 + 3] =
  [a→3, b→2] [c→5] ∨ [a→2, b→3] [c→5] =
  [a→T, b→T, c→5]
- f([a→3, b→2]∨[a→2, b→3]) =
  f([a→T, b→T]) =
  [a→T, b→T] [c→ [a→T, b→T][a] + [a→T, b→T][b]] =
  [a→T, b→T, c→T]

Lack of Distributivity Anomaly



Lack of Distributivity Imprecision:
[a→T, b→T, c→5] more precise.

Summary

- Formal dataflow analysis framework
  - Lattices, partial orders.
  - Transfer functions, joins and splits.
  - Dataflow equations and fixed point solutions.
- Connection with program
  - Abstraction function $AF: \mathbb{S} \to \mathbb{P}$
  - For any state s and program point n, $AF(s) \leq in_n$
  - Meet over paths solutions, distributivity.